

Gessel-Lucas congruences, constant terms, and modular forms

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LSU

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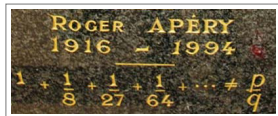
University of South Alabama

THM
Lucas
1878

$$\binom{n}{k} \equiv \binom{n_0}{k_0} \binom{n_1}{k_1} \binom{n_2}{k_2} \cdots \pmod{p}$$

where n_i and k_i are the base p digits of n and k .

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \\ = \text{ct} \left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz} \right)^n \right]$$



Slides available at:

<http://arminstraub.com/talks>

Lucas congruences



THM
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EG

$$\binom{145}{37} \equiv \binom{2}{0} \binom{6}{5} \binom{5}{2} = 1 \cdot 6 \cdot 10 \equiv 4 \pmod{7}$$

$$\text{LHS} = 44141658097075862739392390650979600$$

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- Interesting sequences like the **Apéry numbers**

1, 5, 73, 1445, ...

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy such **Lucas congruences** as well:

THM
Gessel '82

$$A(n) \equiv A(n_0)A(n_1) \cdots A(n_r) \pmod{p}$$

- Equivalently: $A(pn + k) \equiv A(n)A(k) \pmod{p}$

Here and elsewhere: $0 \leq k < p$



Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers**

1, 5, 73, 1445, ...

satisfy

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

THM
Apéry '78

$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.



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THM
Apéry '78

$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the “near”-integers

$$B(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^n \frac{1}{j^3} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \rightarrow \zeta(3)$. But too fast for $\zeta(3)$ to be rational. \square

Apéry numbers and the irrationality of $\zeta(3)$

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THM
Apéry '78

$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

Q

Beukers,
Zagier,
Almkvist,
Zudilin,
Cooper

Are there other tuples (a, b, c) for which the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

has an integral solution?

- Similar (and intertwined) story for:

$$\bullet (n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1} \quad (\text{Beukers, Zagier})$$

$$\bullet (n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1} \quad (\text{Cooper})$$

- 6 + 6 + 3 **sporadic sequences** known.

The six (basic) sporadic Apéry-like numbers of order 3

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}$$

(a, b, c)	$A(n)$	
$(17, 5, 1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12, 4, 16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Kauers–Zeilberger diagonal
$(10, 4, 64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7, 3, 81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11, 5, 125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9, 3, -27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry-like numbers

- Beukers ('87) observed that the Apéry numbers

1, 5, 73, 1145, ...

$$A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$



satisfy:

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geq 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}}$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $q - 12q^2 + 66q^3 + O(q^4)$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:
 - $f(\tau)$ modular form of weight k
 - $x(\tau)$ modular function
 - $y(x)$ such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

- Lucas congruences: $A(pn + k) \equiv A(n)A(k) \pmod{p}$

THM
Malik–S
'16

All of the $6 + 6 + 3$ known sporadic sequences satisfy Lucas congruences modulo every prime. (Proof long and technical for 2 sequences)



Gessel–Lucas congruences

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THM
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- In the case of the Apéry numbers, Gessel ('82) observed that these congruences can be extended modulo p^2 .



THM
S '24

All of the $6 + 6 + 3$ known sporadic sequences satisfy **Gessel–Lucas congruences** modulo every odd prime:

$$A(pn + k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}$$

- Here, $A'(n)$ is the formal derivative of $A(n)$.
These are rational numbers!

The formal derivative of recurrence sequences: example

- $A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}$ is the unique solution with $A(0) = 1$ to:

$$(n+1)^2 A(n+1) = (11n^2 + 11n + 3)A(n) + n^2 A(n-1)$$

- Then $A'(n)$ is the unique solution with $A'(0) = 0$ to:

$$(n+1)^2 A'(n+1) = (11n^2 + 11n + 3)A'(n) + n^2 A'(n-1) \\ - 2(n+1)A(n+1) + 11(2n+1)A(n) + 2nA(n-1)$$

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EG

$$A'(1), A'(2), \dots = 5, \frac{75}{2}, \frac{1855}{6}, \frac{10875}{4}, \frac{299387}{12}, \frac{943397}{4}, \frac{63801107}{28}, \dots$$

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- In this case, $A'(n)$ also is the usual derivative of a natural interpolation:

$$A'(n) = \frac{d}{dx} \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k} \Big|_{x=n} = 5 \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} (H_n - H_k)$$

Approaches to proving Lucas congruences

- From suitable expressions as a **binomial sum**.

Gessel '82, McIntosh '92

$$\text{Apéry numbers: } \sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$$

$$\text{Sequence } (\eta): \sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$$

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- From suitable **constant term** expressions.

Samol-van Straten '09, Mellit-Vlasenko '16

THM
Samol, van
Straten '09

Suppose the origin is the only interior integral point of the Newton polytope of $P \in \mathbb{Z}[x^{\pm 1}]$.

Then $A(n) = \text{ct}[P(x)^n]$ satisfies Lucas congruences.



$$P = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}$$

$$\left(1 - \frac{1}{xy(1+z)^5}\right) \frac{(1+x)(1+y)(1+z)^4}{z^3}$$

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Rowland–Yassawi '15

For instance, diagonals of $1/Q(x)$ for $Q(x) \in \mathbb{Z}[x]$ with $Q(x)$ linear in each variable and $Q(\mathbf{0}) = 1$.

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- From suitable **modular parametrizations**.

Beukers–Tsai–Ye '25

Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its p -truncation.

LEM $A(n)$ satisfies Lucas congruences modulo p

$$\iff \frac{1}{F^{p-1}(x)} \text{ modulo } p \text{ is a polynomial of degree } < p.$$

proof

$$\begin{aligned} A(pn + k) &\equiv A(n)A(k) && \pmod{p} \\ \iff F(x) &\equiv F(x^p)F_p(x) && \pmod{p} \end{aligned}$$



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Since the first p coefficients of \dots always match, the final congruence is equivalent to the RHS being a polynomial of degree $\leq p - 1$. \square

Lucas congruences via modular forms

- Suppose $F(x) = \sum_{n=0}^{\infty} A(n)x^n$ has **modular parametrization**:
 $F(x)$ is a modular form for some modular function $x(\tau)$.

THM
Beukers–
Tsai–Ye
'25

Suppose that:

- $x(\tau) = q + q^2\mathbb{Z}[[q]]$ with $q = e^{2\pi i\tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight 2 modular form for Γ .
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order ≤ 1 , where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

Then $A(n)$ satisfies the Lucas congruences for all primes p .



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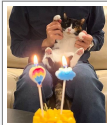
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$$\frac{1}{F^{p-1}(x)} \equiv \quad (\text{mod } p)$$

□



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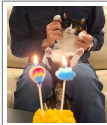
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$$\frac{1}{F^{p-1}(x)} \equiv \frac{E(\tau)}{F^{p-1}(x)} \pmod{p}$$

- $E(\tau)$ is chosen to be a modular form for Γ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$.

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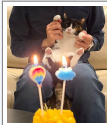
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- $E(\tau)$ is chosen to be a modular form for Γ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$.
- The **modular function** has a unique pole at $[\tau_0]$ of order $\leq p-1$.

□



- Needed: weight $2(p-1)$ modular form $E(\tau)$ for Γ with $E(\tau) \equiv 1 \pmod{p}$.

EG The normalized **Eisenstein series**

$$E_k(\tau) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}$$

is a modular form for $\Gamma_0(1)$ of even weight $k \geq 2$.

Since $1/B_{p-1} \equiv 0 \pmod{p}$, we have $E_{p-1}(\tau) \equiv 1 \pmod{p}$.



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$$E(\tau) := E_{p-1}(\tau)^2$$

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- If $p \geq 5$ and Γ is $\Gamma_0(N)$ extended by $\tau \rightarrow -\frac{1}{N\tau}$:

$$E(\tau) := \frac{1}{2} [E_{p-1}(\tau)^2 + N^{p-1} E_{p-1}(N\tau)^2]$$

The known sporadic sequences satisfy the **Gessel–Lucas congruences**

$$A(pn + k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}.$$

- These generalize:
 - Lucas congruences: $A(pn + k) \equiv A(k)A(n) \pmod{p}$
 - **Supercongruences:** $A(pn) \equiv A(n) \pmod{p^2}$
 - Beukers, Tsai, Ye are working on modular forms proof of Gessel–Lucas congruences.
 - In terms of **linear p -schemes**:
 - Lucas congruences correspond to single-state schemes.
 - Gessel–Lucas congruences are instances of 2-state schemes.
- It would be of interest to study **few-state p -schemes** systematically.
- Are there interesting **q -analog**s?
 - q -Lucas congruences have been studied. Olive '65, Désarménien '82
 - q -analogs known for some supercongruences. S '19, Gorodetsky '19

THANK YOU!

Slides for this talk will be available from my website:

<http://arminstraub.com/talks>



Frits Beukers, Wei-Lun Tsai, Dongxi Ye

Lucas congruences using modular forms

Bulletin of the London Mathematical Society, Vol. 57, 2025, p. 69-78



Joel Henningsen, Armin Straub

Generalized Lucas congruences and linear p -schemes

Advances in Applied Mathematics, Vol. 141, 2022, p. 1-20, #102409



Armin Straub

Gessel-Lucas congruences for sporadic sequences

Monatshefte für Mathematik, Vol. 203, 2024, p. 883–898