Review 14. $\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$

To verify, use $\ln|x| = \begin{cases} \ln(x), & \text{if } x > 0, \\ \ln(-x), & \text{if } x < 0, \end{cases}$ to differentiate the right-hand side.

Substitution, cont'd

In general, finding the right substitution can be tricky and there are no straightforward rules that work for every integral. However, for our integrals there is usually a natural choice.

On the other hand, there is often more than one way to substitute successfully (see next example).

Example 15. (cont'd) Determine $\int x\sqrt{x^2+1}\mathrm{d}x$

(a) by substituting $u = x^2 + 1$, and

(b) by substituting $u = \sqrt{x^2+1}$.

(c) Explain why the substitution $u = x^2 + 1$ is particularly natural.

Solution.

(a) Since $u = x^2 + 1$, we have $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ so that $x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
The integral therefore becomes

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

(b) Since $u = \sqrt{x^2+1}$, we have $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{u}$ so that $x\,\mathrm{d}x = u\mathrm{d}u$.
The integral therefore becomes

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int u \cdot u\,\mathrm{d}u = \int u^2\,\mathrm{d}u = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

(c) The $\sqrt{x^2+1}$ in the integrand will become $\sqrt{u}$ (of course) while the remaining x in the integrand is (up to a factor of 2) exactly the derivative of x^2+1 (and so will disappear when we bring in $\mathrm{d}u$ to replace $\mathrm{d}x$). With a bit of practice, this allows us to immediately see that our substitution will be a success.

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Substitution in definite integrals

If we want to apply substitution in a definite integral like $\int_a^b f(x)\mathrm{d}x$, we have two options:

(a) We can first compute the indefinite integral $\int f(x)\mathrm{d}x$ using substitution.

If the result is $F(x)+C$, then $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$.

(b) Or, we can substitute the limits a and b as well to get a new definite integral $\int_c^d g(u)\mathrm{d}u$.

Often you can choose either approach. But for certain problems it might not be possible to compute the indefinite integral explicitly (so the first approach won't work), yet it can be useful to obtain a substituted new integral using the second approach.

Example 16. Determine $\int_0^1 x\sqrt{x^2+1}\mathrm{d}x$.

Solution. Let us illustrate both of the two approaches just mentioned:

(a) In the previous example, we already worked out that $\int x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{3}(x^2+1)^{3/2}+C$. It follows that

$$\int_0^1 x\sqrt{x^2+1}\mathrm{d}x = \left[\frac{1}{3}(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}2^{3/2} - \frac{1}{3}.$$

(b) As in the previous example, we will substitute $u = x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we get $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. Moreover, if $x = 0$ (lower limit) then $u = x^2 + 1 = 0^2 + 1 = 1$. And if $x = 1$ (upper limit) then $u = x^2+1 = 1^2+1 = 2$. The definite integral therefore becomes

$$\int_0^1 x\sqrt{x^2+1}\mathrm{d}x = \int_1^2 \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \left[\frac{1}{3}u^{3/2}\right]_1^2 = \frac{1}{3}2^{3/2} - \frac{1}{3}.$$

Comment. While it looks like the first approach was much quicker, that's only because we had already computed the indefinite integral in the previous example. In general, if we haven't already done so, the second approach is a bit more direct (but requires us to pay attention to the limits of the integral).