Quiz. The quiz on Thursday will feature a shell method problem similar to the following.

Example 42. (review) Consider the region in the first quadrant bounded by the curves $y=x^2$, $y=2-x^2$ and $x=0$. Use the shell method to find the volume of the solid generated by revolving this region about the y-axis.

Solution. First, make a sketch! Note that the region extends from $x=0$ to $x=1$ (if $x=1$ is not clear from the sketch, you can compute the intersection of $y=x^2$, $y=2-x^2$: $x^2=2-x^2$ simplifies to $x^2=1$ or $x=\pm 1$).

The shell at position x has radius x and height $(2-x^2)-x^2=2-2x^2$. Given a thickness of $\mathrm{d}x$, it therefore has (approximate) volume $2\pi x(2-2x^2)\mathrm{d}x$. Hence, the total volume is

$$\int_0^1 2\pi x(2-2x^2)\mathrm{d}x = 4\pi\int_0^1 (x-x^3)\mathrm{d}x = 4\pi\left[\frac{1}{2}x^2-\frac{1}{4}x^4\right]_0^1 = 4\pi\left(\frac{1}{2}-\frac{1}{4}\right)=\pi.$$

Alternatively. Here is how the disk/washer method looks like for the same problem (in this case either method is feasible because the inverse functions can be readily computed). In this case, we consider horizontal cross-sections from $y=0$ to $y=2$ (see sketch). If $0\leqslant y\leqslant 1$, then we have disks of radius $\sqrt{y}$ ($y=x^2$ is equivalent to $x=\sqrt{y}$). On the other hand, if $1\leqslant y\leqslant 2$, then we have disks of radius $\sqrt{2-y}$ ($y=2-x^2$ is equivalent to $x=\sqrt{2-y}$). Overall, the total volume is again

$$\begin{aligned}\int_0^1 \pi(\sqrt{y})^2\mathrm{d}y + \int_1^2 \pi(\sqrt{2-y})^2\mathrm{d}y &= \pi\int_0^1 y\mathrm{d}y + \pi\int_1^2 (2-y)\mathrm{d}y \\ &= \pi\left[\frac{1}{2}y^2\right]_0^1 + \pi\left[2y-\frac{1}{2}y^2\right]_1^2 = \frac{1}{2}\pi + \pi\left(2-\frac{3}{2}\right) = \pi.\end{aligned}$$

Review. The (arc) length of a general curve $y=f(x)$, for $a\leqslant x\leqslant b$, is given by

$$\int_a^b \sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Example 43. (extra) Using our new technology, compute the circumference of a circle of radius r.

Solution. Of course, the final answer has to be $2\pi r$.

- First, in order to work with a function, we consider the (upper) half circle.

  That half-circle is described by $y = \sqrt{r^2 - x^2}$ and the length of that curve (from $x = -r$ to $x = r$) is half of the circumference of our circle.

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-x}{\sqrt{r^2 - x^2}}$

- Hence, the circumference of our circle is given by the integral

$$2\int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\,\mathrm{d}x = 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,\mathrm{d}x = 2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x.$$

- Now, we substitute $u = x/r$.

  Why? Just looking at the integral, the reason for choosing this substitution might not be obvious. However, thinking about our actual problem, this substitution is very natural: it scales things by $1/r$ so that our circle gets rescaled to a circle of radius 1.

It is very common in applications that we need to change scales or coordinate systems. When dealing with integrals, we then need to perform the corresponding substitution.

  Such changes of scale or coordinate systems for practical reasons is a second important reason why we need to be able to substitute. (So far we substituted as a means to mathematically simplify an integral.)

- To substitute, we compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{r}$, and so $\mathrm{d}x = r\,\mathrm{d}u$ [r is just a number—we can treat it like we would treat 7.]

  For the boundaries of the integral: if $x = -r$, then $u = -1$. If $x = r$, then $u = 1$.

  Since $x = ru$, we therefore get

$$2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x = 2\int_{-1}^{1} \sqrt{\frac{r^2}{r^2 - (ru)^2}} \cdot r\,\mathrm{d}u = 2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\,\mathrm{d}u.$$

- We can actually evaluate this final integral (more on such integrals later) if we recall that the derivative of $\arcsin(u)$ is $1/\sqrt{1 - u^2}$. Hence,

$$2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\,\mathrm{d}u = 2r\Big[\arcsin(u)\Big]_{-1}^{1} = 2r[\arcsin(1) - \arcsin(-1)] = 2\pi r.$$

  Alternatively, substitute $u = \sin(x)$ in the integral.

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Physical work

Work is **force** times **distance**: $W = Fd$.

- F could be measured in lb and d in ft. Then W is conveniently measured in $\mathrm{ft\text{-}lb}$.
- The SI units for F are N (newton), for d they are m (meter), and W is measured in Nm (newtonmeter) or joule ($1\ \mathrm{Nm} = 1\ \mathrm{joule} \approx 0.738\ \mathrm{ft\text{-}lb}$).

Think about the work necessary to lift an object to a certain height.

Calculus comes in when the force F is not constant!

Example 44. Suppose we wish to lift a $100\ \mathrm{lb}$ piano from the ground to the top of a $20\ \mathrm{ft}$ building (for instance, by standing on the roof and pulling it up using a rope). The work required for that is

$$\mathrm{work} = (100\ \mathrm{lb})(20\ \mathrm{ft}) = 2000\,\mathrm{ft\text{-}lb}.$$

This was easy because the force was constant througout the problem (the piano always weighed $100\ \mathrm{lb}$). It is when the force varies (as in the next example) that we need our calculus skills and mastery of integrals.

Example 45. As before, we wish to lift a $100\ \mathrm{lb}$ piano from the ground to the top of a $20\ \mathrm{ft}$ building. We are doing so by standing on the roof and pulling it up using a rope. However, this time, we are using a rather heavy rope weighing $0.1\ \mathrm{lb/ft}$ and want to take that into account (just pulling up the rope, dangling to the ground, would require some work).

Think about the moment when the piano is $x\ \mathrm{ft}$ off the ground (and we want to pull it up by $\mathrm{d}x\ \mathrm{ft}$):

- We still need to pull up $20 - x\ \mathrm{ft}$ so that is the length of the rope at this moment.

  So, at that moment, the weight (piano plus rope) to be pulled up is $100 + 0.1(20 - x)\ \mathrm{lb}$.
- Hence, to pull up the piano by a tiny amount of $\mathrm{d}x$ feet, the amount of work needed is (roughly) $[100 + 0.1(20 - x)]\mathrm{d}x$ pound.

  [Assuming that $\mathrm{d}x$ is very small, the change in weight is insignificant, so that we can use $W = Fd$.]

To get the total amount of work (in $\mathrm{ft\text{-}lb}$), we need to "add" up these small contributions from $x = 0$ to $x = 20$:

$$\mathrm{work} = \int_0^{20} [100 + 0.1(20 - x)]\mathrm{d}x.$$

It only remains to calculate this integral (which is very simple in this case):

$$\mathrm{work} = \int_0^{20} [102 - 0.1x]\mathrm{d}x = \left[102x - \frac{0.1}{2}x^2\right]_0^{20} = 2020\ \mathrm{ft\text{-}lb}.$$

Comment. In this simple example, you can get away with not computing any integrals by arguing as follows: the rope weighs a total of $20 \cdot 0.1 = 2\ \mathrm{lb}$. On average, we need to lift it $10\ \mathrm{ft}$ so that the total amount of work needed to pull up the rope is $2\ \mathrm{lb} \cdot 10\ \mathrm{ft} = 20\,\mathrm{ft\text{-}lb}$. Added to the work required for just the piano ($2000\,\mathrm{ft\text{-}lb}$, see previous example), we get the total that we just computed.

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