

Example 68. (review) Solve the initial value problem $\frac{dy}{dx} = xy$, $y(0) = 3$.

Solution. We separate variables to get $\frac{1}{y} dy = x dx$.

Integrating both sides, we find $\ln(y) = \frac{1}{2}x^2 + C$. (Since $y = 3$ in the initial condition, we don't need to write $\ln|y|$ because we have $y > 0$ around the initial condition.)

We can then find C by using the values $x = 0$, $y = 3$ from the initial condition: $\ln(3) = \frac{1}{2} \cdot 0^2 + C$. So, $C = \ln(3)$.

We now solve $\ln(y) = \frac{1}{2}x^2 + \ln(3)$ for y to get $y(x) = e^{\frac{1}{2}x^2 + \ln(3)} = 3e^{\frac{1}{2}x^2}$.

Alternatively. We could have also first solved for y and then determined C with the same result.

Logarithms

Review. $\ln(x) = \log_e(x)$ is the inverse function of e^x . In other words, for all real x ,

$$\ln(e^x) = x.$$

Similarly, $e^{\ln(x)} = x$ for all $x > 0$.

Likewise. $\log_a(x)$ is the inverse function of a^x (where a is called the base).

- $\frac{d}{dx} \ln(x) = \frac{1}{x}$

Why? Start with $e^{\ln(x)} = x$ and differentiate both sides to get $e^{\ln(x)} \cdot \ln'(x) = 1$. It therefore follows that $\ln'(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$.

Note. We also have $\frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$ for $x < 0$. Together, this implies the next entry.

- $\int \frac{1}{x} dx = \ln|x| + C$

The following observation allows us to convert other bases to the **natural base** e :

(other bases) $a^x = e^{\ln(a)x}$ and $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

Why? $a^x = e^{\ln(a)x}$ follows by writing $a^x = e^{\ln(a^x)} = e^{x \ln(a)}$. Can you see how $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ follows from this?

Alternatively, the defining property of $\log_a$ is that $\log_a(a^x) = x$. Because $\ln(a^x) = x \ln(a)$ we see that $f(x) = \frac{\ln(x)}{\ln(a)}$ also has the property that $f(a^x) = x$.

Example 69. Compute $\frac{d}{dx} \log_a(x)$.

Solution. It follows from $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ that $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$.

Example 70. Compute $\frac{d}{dx} 2^x$ and $\int 2^x dx$.

Solution. Write $2^x = e^{\ln(2^x)} = e^{x \ln(2)}$ to see that $\frac{d}{dx} 2^x = \ln(2) e^{x \ln(2)} = \ln(2) 2^x$ and, likewise, $\int 2^x dx = \frac{1}{\ln(2)} e^{x \ln(2)} + C = \frac{2^x}{\ln(2)} + C$.

Example 71. Rewrite $2^{\ln(x)}$ as a power of x .

Solution. We use the same “trick” as in the previous example (applying $e^{\ln(x)}$ to the expression) to get

$$2^{\ln(x)} = e^{\ln(2^{\ln(x)})} = e^{\ln(x)\ln(2)} = (e^{\ln(x)})^{\ln(2)} = x^{\ln(2)}.$$

Alternatively. Since $\ln(x) = \ln(2)\log_2(x)$, we have $2^{\ln(x)} = 2^{\ln(2)\log_2(x)} = (2^{\log_2(x)})^{\ln(2)} = x^{\ln(2)}$.

Example 72. Determine $\int \frac{2^{\ln(x)}}{x^2} dx$.

Solution. By the previous example,

$$\int \frac{2^{\ln(x)}}{x^2} dx = \int x^{\ln(2)-2} dx = \frac{1}{\ln(2)-1} x^{\ln(2)-1} + C.$$

Example 73. Determine $\int \frac{(\ln x)^4}{3x} dx$.

Solution. We substitute $u = \ln(x)$ in which case $du = \frac{1}{x} dx$ (so that the $\frac{1}{x}$ in the integrand will cancel out), to get

$$\int \frac{(\ln x)^4}{3x} dx = \frac{1}{3} \int u^4 du = \frac{1}{15} u^5 + C = \frac{1}{15} (\ln x)^5 + C.$$

Example 74. Determine $\int \frac{\ln(\ln x)}{x \ln x} dx$.

Solution. We substitute $u = \ln(x)$ in which case $du = \frac{1}{x} dx$ (so that the $\frac{1}{x}$ in the integrand will cancel out). Since

$$\int \frac{\ln(\ln x)}{x \ln x} dx = \int \frac{\ln u}{u} du.$$

In the new integral, we substitute $v = \ln(u)$ with $dv = \frac{1}{u} du$ to get

$$\int \frac{\ln(\ln x)}{x \ln x} dx = \int \frac{\ln u}{u} du = \int v dv = \frac{1}{2} v^2 + C = \frac{1}{2} (\ln u)^2 + C = \frac{1}{2} (\ln(\ln x))^2 + C.$$

Example 75. Determine $\int_0^2 3^{-x} dx$.

Solution. Write $3^{-x} = e^{\ln(3^{-x})} = e^{-x\ln(3)}$ to see that

$$\int_0^2 3^{-x} dx = \left[-\frac{1}{\ln(3)} e^{-x\ln(3)} \right]_0^2 = \left[-\frac{1}{\ln(3)} 3^{-x} \right]_0^2 = \frac{1}{\ln(3)} (1 - 3^{-2}) = \frac{8}{9\ln(3)}.$$

Example 76. Determine $\int \frac{\log_3(x)}{x} dx$.

Solution. Since $\log_3(x) = \frac{\ln(x)}{\ln(3)}$, we find that $\int \frac{\log_3(x)}{x} dx = \frac{1}{\ln(3)} \int \frac{\ln(x)}{x} dx$.

We now substitute $u = \ln(x)$ in which case $du = \frac{1}{x} dx$ (so that the $\frac{1}{x}$ in the integrand will cancel out). Since

$$\int \frac{\ln(x)}{x} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln(x))^2 + C,$$

we get that $\int \frac{\log_3(x)}{x} dx = \frac{1}{\ln(3)} \int \frac{\ln(x)}{x} dx = \frac{1}{2\ln(3)} (\ln(x))^2 + B$ (where $B = \frac{C}{\ln(3)}$ is some constant).