

Improper integrals

Example 114. Determine $\int_0^{\infty} e^{-x} dx$.

This integral is an example of an **improper integral** of type I (because one of its limits is ∞).

Make a sketch!

Solution. Replacing the upper limit with b , we have $\int_0^b e^{-x} dx = [-e^{-x}]_0^b = 1 - e^{-b}$.

Therefore, $\int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} (1 - e^{-b}) = 1$.

Solution. (short version) Once experienced, we can just write $\int_0^{\infty} e^{-x} dx = [-e^{-x}]_0^{\infty} = 0 - (-1) = 1$ with the understanding that we used $\lim_{x \rightarrow \infty} -e^{-x} = 0$.

Example 115. Determine $\int_1^{\infty} \frac{1}{x^4} dx$ as well as $\int_1^{\infty} \frac{1}{x} dx$.

Make a sketch! In the first quadrant, both functions look pretty similar.

Solution.

$$(a) \int_1^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_1^{\infty} = 0 - \left(-\frac{1}{3} \right) = \frac{1}{3} \text{ where we used that } \lim_{x \rightarrow \infty} -\frac{1}{3x^3} = 0.$$

$$(b) \int_1^{\infty} \frac{1}{x} dx = [\ln|x|]_1^{\infty} \text{ but } \lim_{x \rightarrow \infty} \ln|x| = \infty. \text{ We thus say that this integral } \mathbf{diverges} \text{ (to } \infty \text{ in this case).}$$

Example 116. Determine $\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx$.

Make a sketch! (For that, note that the integrand is always positive and that it goes to 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$. Also, you can see that the maximum occurs at $x = 0$. Taken together, the graph looks like a single mound.)

Solution. Recall that $\int \frac{1}{x^2 + 1} dx = \arctan(x) + C$. If necessary, review $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and its inverse function $\arctan(x)$ to recall that

$$\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}, \quad \lim_{x \rightarrow -\infty} \arctan(x) = -\frac{\pi}{2}.$$

$$\text{We therefore have } \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan(x) \right]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi.$$

The following integral is an example of an improper integral of type II (because the integrand has a vertical asymptote at one of the limits).

Example 117. Determine $\int_0^1 \frac{1}{x} dx$.

Make a sketch!

Solution. $\int_0^1 \frac{1}{x} dx = [\ln|x|]_0^1$ but $\lim_{x \rightarrow 0^+} \ln|x| = -\infty$.

Thus, the integral diverges (to ∞ , in this case).

Example 118. The following is VERY WRONG:

$$[\text{bad!}] \quad \int_{-2}^2 \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1} \right]_{-2}^2 = -\frac{1}{3} - 1 = -\frac{4}{3} \quad [\text{bad!}]$$

Note how even the answer is screaming trouble: we integrated something positive and got a negative value.

What went wrong? The integrand has a problem at $x = -1$!

To be precise, it has a vertical asymptote at $x = -1$. Since $\frac{1}{(x+1)^2}$ is not differentiable (not even continuous) at $x = -1$, we can only use the antiderivative $-\frac{1}{x+1}$ for $x \neq -1$. Since -1 is in the domain of integration $[-2, 2]$, we cannot directly apply the Fundamental Theorem of Calculus to this integral.

Instead, we need to split the integral into two improper integrals and analyze these individually:

$$\int_{-2}^2 \frac{dx}{(x+1)^2} = \int_{-2}^{-1} \frac{dx}{(x+1)^2} + \int_{-1}^2 \frac{dx}{(x+1)^2}.$$

But $\int_{-2}^{-1} \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1} \right]_{-2}^{-1}$ diverges because $\lim_{x \rightarrow -1^-} \left(-\frac{1}{x+1} \right) = \infty$. Hence, our original integral diverges.

[Note that, working on the second integral, the limit we encounter is $\lim_{x \rightarrow -1^+} \left(-\frac{1}{x+1} \right) = -\infty$. This integral, by itself, also diverges.]