

Example 119. Do $\int_{1/3}^2 \frac{dx}{\sqrt{3x-1}}$ and $\int_{1/3}^2 \frac{dx}{3x-1}$ converge? If so, compute the integral.

Solution. Note that both integrals have a vertical asymptote at $x = 1/3$, which is the lower limit of integration.

$$(a) \int_{1/3}^2 \frac{dx}{\sqrt{3x-1}} = \left[\frac{2}{3} \sqrt{3x-1} \right]_{1/3}^2 = \frac{2}{3} \sqrt{5} - 0 = \frac{2}{3} \sqrt{5}$$

In particular, the integral converges.

$$(b) \int_{1/3}^2 \frac{dx}{3x-1} = \left[\frac{1}{3} \ln|3x-1| \right]_{1/3}^2 \text{ but } \lim_{x \rightarrow 1/3^+} \frac{1}{3} \ln|3x-1| = -\infty.$$

Thus, the integral diverges (to ∞ , in this case).

Direct comparison and limit comparison

Sometimes we just want to know if an integral converges or diverges. In that case, we can compare the integrand with simpler functions.

The following illustrates our approach but is not meant to be exhaustive. The same ideas apply (suitably adjusted), for instance, when the functions are negative.

We assume that both $f(x)$ and $g(x)$ are continuous on $[a, \infty)$ and positive.

(direct comparison)

- $\int_a^\infty f(x)dx$ converges if $0 \leq f(x) \leq g(x)$ on $[a, \infty)$ and $\int_a^\infty g(x)dx$ converges.
- $\int_a^\infty f(x)dx$ diverges if $f(x) \geq g(x) \geq 0$ on $[a, \infty)$ and $\int_a^\infty g(x)dx$ diverges.

(limit comparison)

- If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ for $0 < L < \infty$ then $\int_a^\infty f(x)dx$ converges if and only if $\int_a^\infty g(x)dx$ converges.

Example 120. Determine whether the following integrals converge or diverge.

$$(a) \int_1^\infty \frac{1}{x^5+2} dx$$

$$(c) \int_2^\infty \frac{\sqrt{x+4}}{x^2} dx$$

$$(f) \int_1^\infty \frac{e^x}{x^2} dx$$

$$(b) \int_2^\infty \frac{\sqrt{x^2+4}}{x^2} dx$$

$$(d) \int_2^\infty \frac{4 - \sin(x)}{x^2} dx$$

$$(g) \int_1^\infty \frac{1}{\sqrt{e^{2x}+3x}} dx$$

$$(e) \int_2^\infty \frac{4 - \sin(x)}{x} dx$$

Solution. The following are just indications of how to proceed. Fill in the details!

- (a) We can apply the limit comparison test with $\frac{1}{x^5+2}$ and $\frac{1}{x^5}$ because $\lim_{x \rightarrow \infty} \frac{1/(x^5+2)}{1/x^5} = 1$.
Since $\int_1^\infty \frac{1}{x^5} dx = \left[-\frac{1}{4x^4} \right]_1^\infty = \frac{1}{4}$ converges, it follows that $\int_1^\infty \frac{1}{x^5+2} dx$ converges as well.
- (b) Do limit comparison with $\frac{\sqrt{x^2}}{x^2} = \frac{1}{x}$ to conclude that this integral diverges.
- (c) Do limit comparison with $\frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$ to conclude that this integral converges.
- (d) Do a direct comparison with $\frac{5}{x^2}$ to conclude that this integral converges.
- (e) Do a direct comparison with $\frac{3}{x}$ to conclude that this integral diverges.
- (f) Note that $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \infty$. Hence the integral obviously diverges.
- (g) Do limit comparison with $\frac{1}{\sqrt{e^{2x}}} = e^{-x}$ to conclude that this integral converges.