

Example 121. Determine whether the following integrals converge or diverge.

$$(a) \int_1^{\infty} \frac{5\sqrt{x} + 4x^2}{x^5 + 3x + 2} dx \quad (b) \int_1^{\infty} \frac{5\sqrt{x} + 2x^4}{x^5 + 3x + 2} dx \quad (c) \int_2^{\infty} \frac{4 - \sin(x)}{x^2} dx$$

Solution.

(a) For large x the integrand behaves like $\frac{4x^2}{x^5} = \frac{4}{x^3}$.

The integral $\int_1^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right]_1^{\infty}$ converges (the exact value doesn't matter) and therefore the original integral converges by limit comparison.

Note. To be precise, we are using that, if $f(x) = \frac{5\sqrt{x} + 4x^2}{x^5 + 3x + 2}$ and $g(x) = \frac{1}{x^3}$, then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{(5\sqrt{x} + 4x^2)x^3}{x^5 + 3x + 2} = \lim_{x \rightarrow \infty} \frac{4x^2 \cdot x^3}{x^5} = 4$$

is a finite number L and $L \neq 0$. You learned about such limits in Calculus I and we will briefly review limits next.

(b) For large x the integrand behaves like $\frac{2x^4}{x^5} = \frac{2}{x}$.

The integral $\int_1^{\infty} \frac{1}{x} dx = [\ln|x|]_1^{\infty}$ diverges (because $\ln|x| \rightarrow \infty$ as $x \rightarrow \infty$) and therefore the original integral diverges by limit comparison.

(c) Note that $\sin(x)$ takes values between -1 and 1 . We therefore have $0 \leq \frac{4 - \sin(x)}{x^2} \leq \frac{3}{x^2}$ for x in $[2, \infty)$.

The integral $\int_2^{\infty} \frac{3}{x^2} dx = \left[-\frac{3}{x} \right]_2^{\infty}$ converges (the exact value doesn't matter) and therefore the original integral converges by direct comparison.

Review: L'Hospital's rule

Example 122. The following example illustrates that limits of the form " $\frac{\infty}{\infty}$ " are completely undetermined. Anything is possible for the actual limit:

- $\lim_{x \rightarrow \infty} \frac{x^2}{x} = \lim_{x \rightarrow \infty} x = \infty$
- $\lim_{x \rightarrow \infty} \frac{x}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$
- $\lim_{x \rightarrow \infty} \frac{x}{3x} = \lim_{x \rightarrow \infty} \frac{1}{3} = \frac{1}{3}$
- $\lim_{x \rightarrow \infty} \frac{x(1 + \sin^2(x))}{x} = \lim_{x \rightarrow \infty} (1 + \sin^2(x))$ This limit does not exist.

Theorem 123. (L'Hospital's rule) If $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = \infty$, then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}.$$

The same conclusion holds if $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 0$.

[It is important to realize that L'Hospital's rule only applies to the undetermined cases " $\frac{\infty}{\infty}$ " and " $\frac{0}{0}$ ".]

Example 124. $\int_0^{\infty} x e^{-3x} dx =$

Your final answer should be $\frac{1}{9}$.

Along the way, you will need the limit $\lim_{x \rightarrow \infty} x e^{-3x} = \lim_{x \rightarrow \infty} \frac{x}{e^{3x}} \stackrel{\text{L'Hospital}}{=} \lim_{x \rightarrow \infty} \frac{1}{3e^{3x}} = 0$.

Sequences

A **sequence**, often denoted $\{a_n\}$, is an infinite list of its **terms** $a_1, a_2, a_3, \dots$

We'll define it precisely later, but one thing we are interested in is the **limit** $\lim_{n \rightarrow \infty} a_n$ (if it exists).

Here are a few first examples of sequences:

- 2, 4, 6, 8, 10, ... (that is, $a_1 = 2, a_2 = 4, \dots$)

This is the sequence $\{a_n\}$ with $a_n = 2n$. Clearly, $\lim_{n \rightarrow \infty} a_n = \infty$.

- 1, -1, 1, -1, 1, ...

This is the sequence $\{a_n\}$ with $a_n = (-1)^{n-1}$. The limit $\lim_{n \rightarrow \infty} a_n$ does not exist.

Comment. Some part of the sequence "goes to" 1 but another part to -1. There is no single value that all terms approach.

- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$

This is the sequence $\{a_n\}$ with $a_n = \frac{1}{2^n}$. Clearly, $\lim_{n \rightarrow \infty} a_n = 0$.

Preview. We will learn later that the **series** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$ also converges and equals 1. Can you maybe already explain why this is the case?

- 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ...

This is the sequence $\{a_n\}$ where a_n consists of the first n (decimal) digits of π . Clearly, $\lim_{n \rightarrow \infty} a_n = \pi$.

- $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$

This is the sequence $\{a_n\}$ with $a_n = \frac{1}{n^2}$. Clearly, $\lim_{n \rightarrow \infty} a_n = 0$.

Preview. We will learn later that the **series** $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$ also converges and equals $\frac{\pi^2}{6}$.

- 1, 1, 2, 3, 5, 8, 13, 21, ...

These are the **Fibonacci numbers** $\{F_n\}$. They are defined **recursively**: $F_n = F_{n-1} + F_{n-2}$ together with the **initial values** $F_1 = 1, F_2 = 1$. Clearly, $\lim_{n \rightarrow \infty} F_n = \infty$.

- $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$

These are quotients of Fibonacci numbers $\{a_n\}$ with $a_n = \frac{F_{n+1}}{F_n}$.

Numerically, 1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, ... Looks like $\lim_{n \rightarrow \infty} a_n$ exists and is about 1.618.

Sequences and series. In a little bit, we will also be interested in **series**. These are infinite sums such as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$. Do not confuse these two!

Confusing sequences and series would be like confusing a function and its definite integral.