Inverses of power series

Review. By the geometric series, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (provided that $|x|<1$).

Example 126. Derive a recursive description of the power series for $y(x) = \frac{1}{1-x-x^2}$.

Solution. Note that $y(x)$ satisfies the "differential" equation $(1-x-x^2)y=1$ of order 0 (as such, we need 0 initial conditions). We can therefore determine a power series solution as we did in Example 105:

Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1-x-x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1=a_0$.
- $n=1$: $0=a_1-a_0$, so that $a_1=a_0=1$.
- $n\geqslant 2$: $0=a_n-a_{n-1}-a_{n-2}$ or, equivalently, $a_n=a_{n-1}+a_{n-2}$.

This is the recursive description of the Fibonacci numbers F_n! In particular $a_n=F_n$.

The first few terms. $\frac{1}{1-x-x^2} = 1+x+2x^2+3x^3+5x^4+8x^5+13x^6+\ldots$

Comment. The function $y(x)$ is said to be a **generating function** for the Fibonacci numbers.

Challenge. Can you rederive Binet's formula from partial fractions and the geometric series?

Example 127. (HW) Derive a recursive description of the power series for $y(x) = \frac{1+7x}{1-x-2x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1+7x = (1-x-2x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - 2\sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1=a_0$.
- $n=1$: $7=a_1-a_0$, so that $a_1=7+a_0=8$.
- $n\geqslant 2$: $0=a_n-a_{n-1}-2a_{n-2}$.

If we prefer, we can rewrite the final recurrence as $a_{n+2}-a_{n+1}-2a_n=0$ for $n\geqslant 0$. The initial conditions are $a_0=1$, $a_1=8$.

Comment. In terms of the recurrence operator N, the recurrence is $(N^2-N-2)a_n=0$.

Comment. As in Example 44, we can solve this recurrence and obtain a Binet-like formula for a_n. In this particular case, we find $a_n = 3\cdot 2^n - 2(-1)^n$.

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Example 128. (extra) For each of the following compute the first few terms of the power series.

(a) $(a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 \ldots)$

(b) $\dfrac{1}{a_0 + a_1 x + a_2 x^2 + \ldots}$

(c) $\dfrac{1}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots}$

Solution.

(a) $a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + O(x^3)$

(b) The answer is $b_0 + b_1 x + \ldots$ with the property that $(a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 \ldots) = 1$.
By the first part, and comparing coefficients, $a_0 b_0 = 1$, $a_0 b_1 + a_1 b_0 = 0$, $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$, ...
Hence, $b_0 = \frac{1}{a_0}$, $b_1 = -\frac{1}{a_0}(a_1 b_0) = -\frac{a_1}{a_0^2}$, $b_2 = -\frac{1}{a_0}(a_1 b_1 + a_2 b_0) = \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}$.

(c) $\dfrac{1}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots} = 1 - x + \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 + \ldots$

Comment. This reflects $\frac{1}{e^x} = e^{-x}$.