

Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t=0$ and $t=L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L -periodic function.

$$\text{In that case, we obtain the Fourier series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{L}\right) + b_n \sin\left(\frac{2\pi n t}{L}\right) \right).$$

(b) We can extend $f(t)$ to an even $2L$ -periodic function.

$$\text{In that case, we obtain the Fourier cosine series } f(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos\left(\frac{\pi n t}{L}\right).$$

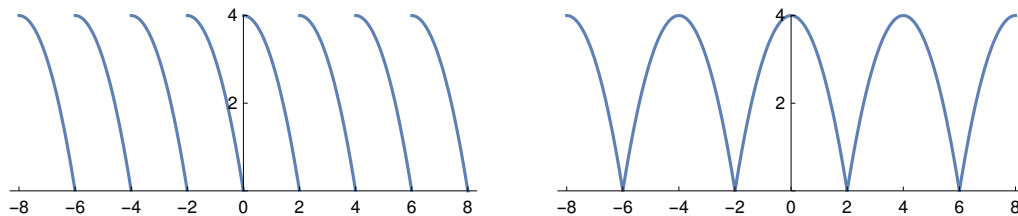
(c) We can extend $f(t)$ to an odd $2L$ -periodic function.

$$\text{In that case, we obtain the Fourier sine series } f(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{\pi n t}{L}\right).$$

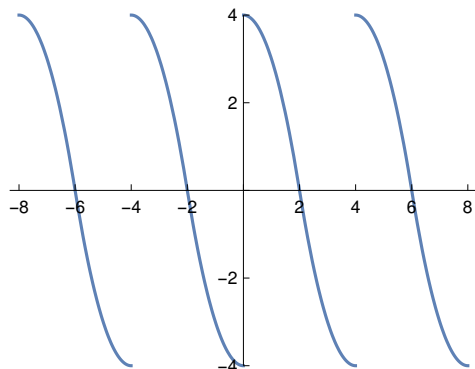
Example 136. Consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

- Sketch the 2-periodic extension of $f(t)$.
- Sketch the 4-periodic even extension of $f(t)$.
- Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:



The 4-periodic odd extension:



Example 137. As in the previous example, consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

- (a) Let $F(t)$ be the Fourier series of $f(t)$ (meaning the 2-periodic extension of $f(t)$). Determine $F(2)$, $F\left(\frac{5}{2}\right)$ and $F\left(-\frac{1}{2}\right)$.
- (b) Let $G(t)$ be the Fourier cosine series of $f(t)$. Determine $G(2)$, $G\left(\frac{5}{2}\right)$ and $G\left(-\frac{1}{2}\right)$.
- (c) Let $H(t)$ be the Fourier sine series of $f(t)$. Determine $H(2)$, $H\left(\frac{5}{2}\right)$ and $H\left(-\frac{1}{2}\right)$.

Solution.

- (a) Note that the extension of $f(t)$ has discontinuities at $\dots, -2, 0, 2, 4, \dots$ (see plot in previous example) and recall that the Fourier series takes average values at these discontinuities:

$$F(2) = \frac{1}{2}(F(2^-) + F(2^+)) = \frac{1}{2}(0 + 4) = 2$$

$$F\left(\frac{5}{2}\right) = F\left(\frac{5}{2} - 2\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$$

$$F\left(-\frac{1}{2}\right) = F\left(-\frac{1}{2} + 2\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$$

- (b) $G(2) = f(2) = 0$ (see plot!)

[Note that $G(2^+) = G(2^+ - 4) = G(-2^+) = G(2^-)$ where we used that G is even in the last step; in fact, we can show like this that the Fourier cosine series of a continuous function is always continuous.]

$$G\left(\frac{5}{2}\right) = G\left(\frac{5}{2} - 4\right) = G\left(-\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$$

$$G\left(-\frac{1}{2}\right) \stackrel{\text{even}}{=} G\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$$

- (c) $H(2) = \frac{1}{2}(f(2^-) - f(2^+)) = 0$ (see plot!)

[Note that $H(2^+) = H(2^+ - 4) = H(-2^+) = -H(2^-)$ where we used that H is odd in the last step; in fact, we can show like this that the Fourier sine series of a continuous function is always 0 at the jumps.]

$$H\left(\frac{5}{2}\right) = H\left(\frac{5}{2} - 4\right) = H\left(-\frac{3}{2}\right) = -f\left(\frac{3}{2}\right) = -\frac{7}{4}$$

$$H\left(-\frac{1}{2}\right) \stackrel{\text{odd}}{=} -H\left(\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) = -\frac{15}{4}$$