Differentiating and integrating Fourier series

Theorem 138. If $f(t)$ is **continuous** and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Caution! We cannot simply differentiate termwise if $f(t)$ is lacking continuity. See the next example.

Comment. On the other hand, we can integrate termwise (going from the Fourier series of $f' = g$ to the Fourier series of $f = \int g$ because the latter will be continuous). This is illustrated in the example after the next.

Example 139. (caution!) The function $g(t) = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(n\pi t)$ from Example 135 is not continuous. For all values, except the discontinuities, we have $g'(t) = 0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n \text{ odd}} \cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t = 0$). This illustrates that we cannot apply Theorem 138 because $g(t)$ is lacking continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

Example 140. Let $h(t)$ be the 2-periodic function with $h(t) = |t|$ for $t \in [-1, 1]$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n = 0$. Computing a_n is left as an exercise.

Solution. Note that $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$ is continuous and $h'(t) = g(t)$, with $g(t)$ as in Example 135. Hence, we can apply Theorem 138 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t) \quad \Longrightarrow \quad h(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \left(-\frac{1}{\pi n}\right) \cos(n\pi t) + C,$$

where $C = \frac{a_0}{2} = \frac{1}{2}\int_{-1}^{1} h(t)\mathrm{d}t = \frac{1}{2}$ is the constant of integration. Thus, $h(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(n\pi t)$.

Remark. Note that $t = 0$ in the last Fourier series, gives us $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$. As an exercise, you can try to find from here the fact that $\sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

Review: the motion of a mass on a spring

The motion of a mass m attached to a spring is described by

$$my'' + ky = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Hooke's law $F = -ky$ combined with Newton's second law $F = ma = my''$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

- The motion $y(t)$ is periodic with **period** $2\pi/\omega$. Equivalently, its (circular) **frequency** is ω.

  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ (can you explain/prove this?) where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$.

More generally, the motion of a mass m on a spring, with damping and with an external force $f(t)$ taken into account, can be modeled by the DE

$$my'' + dy' + ky = f(t).$$

Note that each term is representing a force: $my'' = ma$ is the force due to Newton's second law ($F = ma$), the term dy' models damping (proportional to the velocity), the term ky represents the force due to Hooke's law, and the term $f(t)$ represents an external force that acts on the mass at time t.

Fourier series and linear differential equations

In the following examples, we consider inhomogeneous linear DEs $p(D)y = F(t)$ where $F(t)$ is a periodic function that can be expressed as a Fourier series. We first review the notion of **resonance** (and how to predict it) and then solve such DEs.

Example 141. Consider the linear DE $my'' + ky = \cos(\omega t)$. For which (external) **frequencies** $\omega > 0$ does **resonance** occur?

Solution. The characteristic roots (the roots of $p(D) = mD^2 + k$) are $\pm i\sqrt{k/m}$. Correspondingly, the solutions of the homogeneous equation $my'' + ky = 0$ are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$ (ω_0 is called the **natural frequency** of the DE). Resonance occurs in the case $\omega = \omega_0$ when the external frequency matches the natural frequency.

Review. If $\omega \neq \omega_0$ (overlapping roots), then there is particular solution of the form $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ (for specific values of A and B). The general solution is $y(t) = A\cos(\omega t) + B\sin(\omega t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, which is a bounded function of t. In contrast, if $\omega = \omega_0$, then the general solution is $y(t) = (C_1 + At)\cos(\omega_0 t) + (C_2 + Bt)\sin(\omega_0 t)$ and this function is unbounded.