Example 142. A mass-spring system is described by the DE $2y'' + 32y = \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2+1}$.

For which ω does resonance occur?

Solution. The roots of $p(D) = 2D^2 + 32$ are $\pm 4i$, so that that the natural frequency is 4. Resonance therefore occurs if 4 equals $n\omega$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $\omega = 4/n$ for some $n \in \{1, 2, 3, \ldots\}$.

Example 143. A mass-spring system is described by the DE $my'' + y = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right)$.

For which m does resonance occur?

Solution. The roots of $p(D) = mD^2 + 1$ are $\pm i/\sqrt{m}$, so that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m} = n/3$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $m = 9/n^2$ for some $n \in \{1, 2, 3, \ldots\}$.

Example 144. A mass-spring system is described by the DE $3y'' + ky = F(t)$ where $F(t)$ is an external force with period 5. For which values of k can resonance occur?

Solution. $F(t)$ has a Fourier series of the form $F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{5}\right) + b_n \sin\left(\frac{2\pi n t}{5}\right)\right)$.

The roots of $p(D) = 3D^2 + k$ are $\pm i\sqrt{\frac{k}{3}}$, so that the natural frequency is $\sqrt{\frac{k}{3}}$. Resonance therefore can occur if $\sqrt{\frac{k}{3}} = \frac{2\pi n}{5}$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance can occur if $k = \frac{12\pi^2 n^2}{25}$ for some $n \in \{1, 2, 3, \ldots\}$.

Note. Resonance will occur for $k = \frac{12\pi^2 n^2}{25}$ unless both of the corresponding Fourier coefficients a_n and b_n are 0.

Note. The term $a_0/2$ in $F(t)$ corresponds to a characteristic root of 0 and cannot lead to resonance.

Though it requires some effort, we already know how to solve $p(D)y = F(t)$ for periodic forces $F(t)$, once we have a Fourier series for $F(t)$.

The same approach works for linear differential equations of higher order, or even systems of equations.

Example 145. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution.

- From earlier, we already know $F(t) = 10 \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(\pi n t)$.
- We next solve the equation $2y'' + 32y = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $y_p(t) = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $y_p'' = -\pi^2 n^2 y_p$ (can you see why without computing two derivatives?), we get

$$2y_p'' + 32y_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \overset{!}{=} \sin(\pi n t).$$

  We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $y_p(t) = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.
- We combine the particular solutions found in the previous step, to see that

$$2y'' + 32y = 10 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\pi n t) \quad \text{is solved by} \quad y_p = 10 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

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Example 146. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t)$ the 2π-periodic function such that $F(t) = 10t$ for $t \in (-\pi, \pi)$.

Solution.

- The Fourier series of $F(t)$ is $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$. [Exercise!]
- We next solve the equation $2y'' + 32y = \sin(nt)$ for $n = 1, 2, 3, \ldots$. Note, however, that **resonance** occurs for $n = 4$, so we need to treat that case separately. If $n \neq 4$ then we find, as in the previous example, that $y_p(t) = \frac{\sin(nt)}{32 - 2n^2}$. [Note how this fails for $n = 4$!]

  For $2y'' + 32y = \sin(4t)$, we begin with $y_p = At\cos(4t) + Bt\sin(4t)$. Then $y_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$, and $y_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$. Plugging into the DE, we get $2y_p'' + 32y_p = 16B\cos(4t) - 16A\sin(4t) \stackrel{!}{=} \sin(4t)$, and thus $B = 0$, $A = -\frac{1}{16}$. So, $y_p = -\frac{1}{16}t\cos(4t)$.
- We combine the particular solutions to get that our DE

$$2y'' + 32y = -5\sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$$

  is solved by

$$y_p(t) = \frac{5}{16} t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \frac{\sin(nt)}{32 - 2n^2}.$$

  As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

Important comment. Note that the general solution is

$$y(t) = \frac{5}{16} t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \frac{\sin(nt)}{32 - 2n^2} + C_1\cos(4t) + C_2\sin(4t)$$

and that it always features the resonant term.