Partial differential equations

The heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x,t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x \in [0,L]$.

Notation. $u(x,t)$ depends on two variables. When taking derivatives, we will use the notations $u_t = \frac{\partial}{\partial t}u$ and $u_{xx} = \frac{\partial^2}{\partial x^2}u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx} < 0$) of that profile to flatten out (which means that $u_t < 0$); similarly, minima (where $u_{xx} > 0$) should go up (meaning that $u_t > 0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t = k u_{xx}$, with $k > 0$.

(heat equation)

$$u_t = k u_{xx}$$

Note that the heat equation is a linear and homogeneous **partial differential equation**.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1 u_1 + c_2 u_2$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$. The heat equation is often written as $u_t = k\Delta u$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator you may know from Calculus III.

The Laplacian Δu is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial/\partial x, \partial/\partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla \cdot \nabla$.

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t = 0$: $u(x,0) = f(x)$ (IC)

  This specifies an initial temperature distribution at time $t = 0$.

- **Boundary condition** at $x = 0$ and $x = L$: (BC)

  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:

  - $u(0,t) = A$, $u(L,t) = B$

    This models a rod where one end is kept at temperature A and the other end at temperature B.

  - $u_x(0,t) = u_x(L,t) = 0$

    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Important comment. We can always transform the case $u(0,t) = A$, $u(L,t) = B$ into $u(0,t) = u(L,t) = 0$ by using the fact that $u(t,x) = ax + b$ solves $u_t = k u_{xx}$. Can you spell this out?

Example 160. To get a feeling, let us find some solutions to $u_t = k u_{xx}$.

- $u(x,t) = ax + b$ is a solution.
- For instance, $u(x,t) = e^{kt} e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- More interesting are $u(x,t) = e^{-kt} \cos(x)$ and $u(x,t) = e^{-kt} \sin(x)$.
- More generally, $e^{-k\lambda^2 t} \cos(\lambda x)$ and $e^{-k\lambda^2 t} \sin(\lambda x)$ are solutions.
- Can you find further solutions?

Important observation. This reveals a strategy for solving the heat equation together with the following boundary and initial conditions:

$$\begin{array}{ll} u_t = k u_{xx} & \text{(PDE)} \\ u(0,t) = u(L,t) = 0 & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,L) & \text{(IC)} \end{array}$$

Note that $e^{-k\lambda^2 t} \sin(\lambda x)$ solves the PDE and also satisfies (BC) if $\lambda = n\frac{\pi}{L}$ for some integer n. Hence,

$$u_n(x,t) = e^{-k\left(\frac{\pi n}{L}\right)^2 t} \sin\left(\frac{\pi n}{L} x\right)$$

satisfies the PDE as well as (BC) for any integer n.

It remains to satisfy (IC) and we plan to do so by taking the right combination of the $u_n(x,t)$. At $t = 0$, we get $u_n(x,0) = \sin\left(\frac{\pi n}{L} x\right)$ and all of these are $2L$-periodic and odd. This matches exactly the terms we get when we write $f(x)$ as a Fourier sine series ($f(x)$ is only given on $(0, L)$ and we extend it to an odd $2L$-periodic function):

$$f(x) = \sum_{n \geqslant 1} b_n \sin\left(\frac{\pi n}{L} x\right)$$

Consequently, $(\text{PDE})+(\text{BC})+(\text{IC})$ is solved by

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\pi n}{L}\right)^2 kt} \sin\left(\frac{\pi n}{L} x\right).$$

Comment. Note that the coefficients b_n can be computed as

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Comment. Note that $n = 0$ just gives the zero function $u_0(x,t) = 0$, and negative values don't give anything new because $u_{-n}(x,t) = -u_n(x,t)$.

Example 161. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} &u_t = u_{xx} && \text{(PDE)} \\ &u(0,t) = u(\pi,t) = 0 && \text{(BC)} \\ &u(x,0) = \sin(2x) - 7\sin(3x), \quad x \in (0,\pi) && \text{(IC)} \end{aligned}$$

Solution. This is the case $k = 1$, $L = \pi$ of the above. Hence, as we just observed, the functions

$$u_n(x,t) = e^{-n^2 t}\sin(nx)$$

satisfy (PDE) and (BC) for any integer n.
Since $u_n(x,0) = \sin(nx)$, we have

$$u_2(x,0) - 7u_3(x,0) = \sin(2x) - 7\sin(3x)$$

as needed for (IC).
Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = u_2(x,t) - 7u_3(x,t) = e^{-4t}\sin(2x) - 7e^{-9t}\sin(3x).$$

Example 162. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} &u_t = 3u_{xx} && \text{(PDE)} \\ &u(0,t) = u(4,t) = 0 && \text{(BC)} \\ &u(x,0) = 5\sin(\pi x) - \sin(3\pi x), \quad x \in (0,4) && \text{(IC)} \end{aligned}$$

Solution. This is the case $k = 3$, $L = 4$ of the above. Hence, the functions

$$u_n(x,t) = e^{-3\left(\frac{\pi n}{4}\right)^2 t}\sin\left(\frac{\pi n}{4}x\right)$$

satisfy (PDE) and (BC) for any integer n. Since $u_n(x,0) = \sin\left(\frac{\pi n}{4}x\right)$, we have

$$5u_4(x,0) - u_{12}(x,0) = 5\sin(\pi x) - \sin(3\pi x),$$

which is what we need for the right-hand side of (IC). Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = 5u_4(x,t) - u_{12}(x,t) = 5e^{-3\pi^2 t}\sin(\pi x) - e^{-27\pi^2 t}\sin(3\pi x).$$

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