

### The inhomogeneous heat equation

We next indicate that we can similarly solve the inhomogeneous heat equation (with inhomogeneous boundary conditions).

**Comment.** We indicated earlier that

$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u(0, t) &= a, \quad u(L, t) = b && \text{(BC)} \\ u(x, 0) &= f(x), \quad x \in (0, L) && \text{(IC)} \end{aligned}$$

can be solved by realizing that  $Ax + B$  solves (PDE).

Indeed, let  $v(x) = a + \frac{b-a}{L}x$  (so that  $v(0) = a$  and  $v(L) = b$ ). We then look for a solution of the form  $u(x, t) = v(x) + w(x, t)$ . Note that  $u(x, t)$  solves (PDE)+(BC)+(IC) if and only if  $w(x, t)$  solves:

$$\begin{aligned} w_t &= k w_{xx} && \text{(PDE)} \\ w(0, t) &= 0, \quad w(L, t) = 0 && \text{(BC*)} \\ w(x, 0) &= f(x) - v(x), \quad x \in (0, L) && \text{(IC)} \end{aligned}$$

This is the (homogeneous) heat equation that we know how to solve.

$v(x)$  is called the **steady-state solution** (it does not depend on time!) and  $w(x, t)$  the **transient solution** (note that  $w(x, t)$  and its partial derivatives tend to zero as  $t \rightarrow \infty$  because of the boundary conditions (BC\*)).

**Example 167.** Consider the heat flow problem: 
$$\begin{aligned} u_t &= 3u_{xx} + 4x^2 && \text{(PDE)} \\ u(0, t) &= 1, \quad u_x(3, t) = -5 && \text{(BC)} \\ u(x, 0) &= f(x), \quad x \in (0, 3) && \text{(IC)} \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

**Solution.** We look for a solution of the form  $u(x, t) = v(x) + w(x, t)$ , where  $v(x)$  is the steady-state solution and where  $w(x, t)$  is the transient solution which (together with its derivatives) tends to zero as  $t \rightarrow \infty$ .

- Plugging into (PDE), we get  $w_t = 3v'' + 3w_{xx} + 4x^2$ . Letting  $t \rightarrow \infty$ , this becomes  $0 = 3v'' + 4x^2$ . Note that this also implies that  $w_t = 3w_{xx}$ .
- Plugging into (BC), we get  $v(0) + w(0, t) = 1$  and  $v'(3) + w_x(3, t) = -5$ . Letting  $t \rightarrow \infty$ , these become  $v(0) = 1$  and  $v'(3) = -5$ .
- Solving the ODE  $0 = 3v'' + 4x^2$ , we find

$$v(x) = \iint -\frac{4}{3}x^2 dx dx = \int \left( -\frac{4}{9}x^3 + C \right) dx = -\frac{1}{9}x^4 + Cx + D.$$

The boundary conditions  $v(0) = 1$  and  $v'(3) = -5$  imply  $D = 1$  and  $-\frac{4}{9} \cdot 3^3 + C = -5$  (so that  $C = 7$ ). In conclusion, the steady-state solution is  $v(x) = -\frac{1}{9}x^4 + 1 + 7x$ .

On the other hand, the transient solution  $w(x, t)$  is characterized as the unique solution to:

$$\begin{aligned} w_t &= 3w_{xx} && \text{(PDE*)} \\ w(0, t) &= 0, \quad w_x(3, t) = 0 && \text{(BC*)} \\ w(x, 0) &= f(x) - v(x) && \text{(IC*)} \end{aligned}$$

This homogeneous heat flow problem can now be solved using separation of variables.

**Example 168.** For  $t \geq 0$  and  $x \in [0, 4]$ , consider the heat flow problem:

$$\begin{aligned} u_t &= 2u_{xx} + e^{-x/2} \\ u_x(0, t) &= 3 \\ u(4, t) &= -2 \\ u(x, 0) &= f(x) \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

**Solution.** We look for a solution of the form  $u(x, t) = v(x) + w(x, t)$ , where  $v(x)$  is the steady-state solution and where the transient solution  $w(x, t)$  tends to zero as  $t \rightarrow \infty$  (as do its derivatives).

- Plugging into (PDE), we get  $w_t = 2w_{xx} + e^{-x/2}$ . Letting  $t \rightarrow \infty$ , this becomes  $0 = 2v'' + e^{-x/2}$ .
- Plugging into (BC), we get  $w_x(0, t) + v'(0) = 3$  and  $w(4, t) + v(4) = -2$ .  
Letting  $t \rightarrow \infty$ , these become  $v'(0) = 3$  and  $v(4) = -2$ .
- Solving the ODE  $0 = 2v'' + e^{-x/2}$ , we find

$$v(x) = \iint -\frac{1}{2}e^{-x/2} dx dx = \int (e^{-x/2} + C) dx = -2e^{-x/2} + Cx + D.$$

The boundary conditions  $v'(0) = 3$  and  $v(4) = -2$  imply  $C = 2$  and  $-2e^{-2} + 8 + D = -2$ .

In conclusion, the steady-state solution is  $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$ .

On the other hand, the transient solution  $w(x, t)$  is characterized as the unique solution to:

$$\begin{aligned} w_t &= 2w_{xx} \\ w_x(0, t) &= 0, \quad w(4, t) = 0 \\ w(x, 0) &= f(x) - v(x) \end{aligned}$$

**Note.** We know how to solve this homogeneous heat equation using separation of variables.