Notes for Lecture 35

Wed, 4/22/2026

Steady-state temperature: The Laplace equation

(2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

The heat equation is often written as $u_t = k\Delta u$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (2D) or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (3D) is the **Laplace operator** you may know from Calculus III.

Other notations. $\Delta u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = \nabla^2 u$

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature $u(x, y)$ must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation, 2D)

$$u_{xx} + u_{yy} = 0$$

Comment. The Laplace equation is so important that its solutions have their own name: **harmonic functions**. It is also known as the "potential equation"; satisfied by electric/gravitational potential functions. (More generally, such potentials, if not in the vacuum, satisfy the **Poisson equation** $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.)

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential f).

The divergence of a vector field $\boldsymbol{G} = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is $\operatorname{div} \boldsymbol{G} = g_x + h_y$. One also writes $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

One way to describe a unique solution to the Laplace equation within a region is by specifying the values of $u(x, y)$ along the boundary of that region.

This is particularly natural for steady-state temperatures profiles of a region R. The Laplace equation governs how temperature behaves inside the region but we need to also prescribe the temperature on the boundary.

The PDE with such a boundary condition is called a Dirichlet problem:

(Dirichlet problem)

$u_{xx} + u_{yy} = 0$ within region R

$u(x, y) = f(x, y)$ on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

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Finite difference method: A glance at discretizing PDEs

We know from Calculus that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

PDEs quickly become impossibly difficult to approach with exact solution techniques.

It is common therefore to proceed numerically. One approach is to discretize the problem.

For instance. We could use $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$ to replace $f'(x)$ with the **finite difference** on the RHS.

Such approximate methods are called **finite difference methods**.

Finite difference methods are a common approach to numerically solving PDEs.
The ODE or PDE translates into a (sparse) system of linear equations which is then solved using Linear Algebra.

Example 169.

- $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$ is a **forward difference** for $f'(x)$.
- $f'(x) \approx \frac{1}{h}[f(x) - f(x-h)]$ is a **backward difference** for $f'(x)$.
- $f'(x) \approx \frac{1}{2h}[f(x+h) - f(x-h)]$ is a **central difference** for $f'(x)$.
  Note that this is the average of the forward and the backward difference. The calculations below show that the central difference performs better as an approximation of $f'(x)$.

Comment. Recall that power series $f(x)$ have the Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.

Equivalently, $f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)$. It follows that

$$\frac{1}{h}[f(x+h) - f(x)] = f'(x) + \boxed{\frac{h}{2} f''(x) + O(h^2)} = f'(x) + \boxed{O(h)}.$$

The $\boxed{\text{error}}$ is of order $O(h)$. On the other hand, combining

$$\begin{aligned} f(x+h) &= f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4), \\ f(x-h) &= f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + O(h^4), \end{aligned}$$

it follows that

$$\frac{1}{2h}[f(x+h) - f(x-h)] = f'(x) + \boxed{\frac{h^2}{6} f'''(x) + O(h^3)} = f'(x) + \boxed{O(h^2)}.$$

The $\boxed{\text{error}}$ is of order $O(h^2)$.

Comment. An error of order h^2 means that if we cut h by a factor of, say, $\frac{1}{10}$, then we expect the error to be cut by a factor of $\frac{1}{10^2} = \frac{1}{100}$.

Example 170. Find a central difference for $f''(x)$.

Solution. Adding the two expansions for $f(x+h)$ and $f(x-h)$ to kill the $f'(x)$ terms, and subtracting $2f(x)$, we find that

$$\frac{1}{h^2}[f(x+h)-2f(x)+f(x-h)] = f''(x) + \boxed{\frac{h^2}{12}f^{(4)}(x)+O(h^3)} = f''(x) + \boxed{O(h^2)}.$$

The $\boxed{\text{error}}$ is of order 2.

Alternatively. If we iterate the approximation $f'(x) \approx \frac{1}{2h}[f(x+h)-f(x-h)]$ (in the second step, we apply it with x replaced by $x \pm h$), we obtain

$$f''(x) \approx \frac{1}{2h}[f'(x+h)-f'(x-h)] \approx \frac{1}{4h^2}[f(x+2h)-2f(x)+f(x-2h)],$$

which is the same as what we found above, just with h replaced by $2h$.

Example 171. (discretizing Δ) Use the above central difference approximation for second derivatives to derive a finite difference for $\Delta u = u_{xx}+u_{yy}$ in 2D.

Solution.

$$\begin{aligned} u_{xx} + u_{yy} &\approx \frac{1}{h^2}[u(x+h,y)-2u(x,y)+u(x-h,y)] + \frac{1}{h^2}[u(x,y+h)-2u(x,y)+u(x,y-h)] \\ &= \frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)] \end{aligned}$$

Notation. This finite difference is often represented as $\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$, the **five-point stencil**.

Comment. Recall that solutions to $\Delta u = 0$ are supposed to describe steady-state temperature distributions. We can see from our discretization that this is reasonable. Namely, $\Delta u = 0$ becomes approximately equivalent to

$$u(x,y) = \frac{1}{4}(u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)).$$

In other words, the temperature $u(x,y)$ at a point (x,y) should be the average of the temperatures of its four “neighbors” $u(x+h,y)$ (right), $u(x-h,y)$ (left), $u(x,y+h)$ (top), $u(x,y-h)$ (bottom).

Comment. Think about how to use this finite difference to numerically solve the corresponding Dirichlet problem by discretizing (one equation per lattice point).

Advanced comment. If $\Delta u = 0$ then, when discretizing, the center point has the average value of the four points adjacent to it. This leads to the **maximum principle**: if $\Delta u = 0$ on a region R, then the maximum (and, likewise, minimum) value of u must occur at a boundary point of R.