

**Example 172.** Discretize the following Dirichlet problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{(PDE)} \\ u(x, 0) &= 2 \\ u(x, 2) &= 3 \\ u(0, y) &= 0 \\ u(1, y) &= 0 \end{aligned} \quad \text{(BC)}$$

Use a step size of  $h = \frac{1}{3}$ .

**Comment.** Note that, for the Dirichlet problem as well as for our discretization, it doesn't matter that the boundary conditions aren't well-defined at the corners.

**Solution.** Note that our rectangle has side lengths 1 (in  $x$  direction) and 2 (in  $y$  direction). With a step size of  $h = \frac{1}{3}$  we therefore get  $4 \cdot 7$  lattice points, namely the points

$$u_{m,n} = u(mh, nh), \quad m \in \{0, 1, 2, 3\}, \quad n \in \{0, 1, \dots, 6\}.$$

Further note that the boundary conditions determine the values of  $u_{m,n}$  if  $m = 0$  or  $m = 3$  as well as if  $n = 0$  or  $n = 6$ . This leaves  $2 \cdot 5 = 10$  points at which we need to determine the value of  $u_{m,n}$ .

Next, we approximate  $u_{xx} + u_{yy}$  by  $\frac{1}{h^2}[u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)]$  (see previous example for how we obtained this finite difference approximation). Note that, if  $u(x, y) = u_{m,n}$  is one of our lattice points, then the other four terms in the finite difference are lattice points as well; for instance,  $u(x+h, y) = u_{m+1,n}$ . The equation  $u_{xx} + u_{yy} = 0$  therefore translates into

$$u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0.$$

Spelling out these equation for each  $m \in \{1, 2\}$  and  $n \in \{1, 2, \dots, 5\}$ , we get 10 (linear) equations for our 10 unknown values. For instance, here are the equations for  $(m, n) = (1, 1), (1, 2)$  as well as  $(2, 5)$ :

$$\begin{aligned} u_{2,1} + \underbrace{u_{0,1}}_{=0} + u_{1,2} + \underbrace{u_{1,0}}_{=2} - 4u_{1,1} &= 0 \\ u_{2,2} + \underbrace{u_{0,2}}_{=0} + u_{1,3} + u_{1,1} - 4u_{1,2} &= 0 \\ &\vdots \\ \underbrace{u_{3,5}}_{=0} + u_{1,5} + \underbrace{u_{2,6}}_{=3} + u_{2,4} - 4u_{2,5} &= 0 \end{aligned}$$

In matrix-vector form, these linear equations take the form:

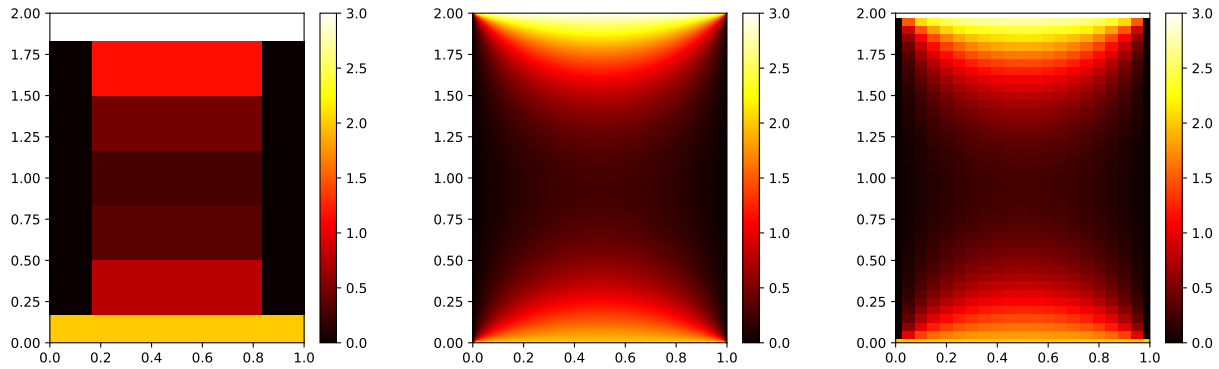
$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,5} \\ u_{2,1} \\ u_{2,2} \\ \vdots \\ u_{2,5} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ \vdots \\ -3 \end{bmatrix}$$

Solving this system, we find  $u_{1,1} \approx 0.7847, u_{1,2} \approx 0.3542, \dots, u_{2,5} \approx 1.1597$ .

For comparison, the corresponding exact values are  $u\left(\frac{1}{3}, \frac{1}{3}\right) \approx 0.7872, u\left(\frac{1}{3}, \frac{2}{3}\right) \approx 0.3209, \dots, u\left(\frac{2}{3}, \frac{5}{3}\right) \approx 1.1679$ . These were computed from the exact formula (derived using separation of variables)

$$u(x, y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{\sin(\pi n x)}{1 - e^{4\pi n}} [2(e^{\pi n y} - e^{-\pi n(y-4)}) + 3(e^{\pi n(2-y)} - e^{\pi n(2+y)})],$$

The three plots below visualize the discretized solution with  $h = \frac{1}{3}$  from Example 172, the exact solution, as well as the discretized solution with  $h = \frac{1}{20}$ .



**Comment.** The first plot looks a bit overly rough because we chose not to interpolate the values. As we showed above, the approximate values at the ten lattice points are actually pretty decent for such a large step size.

**Warning.** The resulting linear systems quickly become very large. For instance, if we use a step size of  $h = \frac{1}{100}$ , then we need to determine roughly  $100 \cdot 200 = 20,000$  ( $99 \cdot 199$  to be exact) values  $u_{m,n}$ . The corresponding matrix  $M$  will have about  $20,000^2 = 400,000,000$  entries, which is already challenging for a weak machine if we use generic linear algebra software. At this point it is important to realize that most entries of the matrix  $M$  are 0. Such matrices are called **sparse** and there are efficient algorithms for solving systems involving such matrices.

**Example 173.** Discretize the following Dirichlet problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{(PDE)} \\ u(x, 0) &= 2 \\ u(x, 1) &= 3 \\ u(0, y) &= 1 \\ u(2, y) &= 4 \end{aligned} \quad \text{(BC)}$$

Use a step size of  $h = \frac{1}{2}$ .

**Solution.** Note that our rectangle has side lengths 2 (in  $x$  direction) and 1 (in  $y$  direction). With a step size of  $h = \frac{1}{2}$  we therefore get  $5 \cdot 3$  lattice points, namely the points

$$u_{m,n} = u(mh, nh), \quad m \in \{0, 1, 2, 3, 4\}, \quad n \in \{0, 1, 2\}.$$

Further note that the boundary conditions determine the values of  $u_{m,n}$  if  $m = 0$  or  $m = 4$  as well as if  $n = 0$  or  $n = 2$ . This leaves  $3 \cdot 1 = 3$  points at which we need to determine the value of  $u_{m,n}$ .

If we approximate  $u_{xx} + u_{yy}$  by  $\frac{1}{h^2}[u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)]$  then, in terms of our lattice points, the equation  $u_{xx} + u_{yy} = 0$  translates into

$$u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0.$$

Spelling out these equation for each  $m \in \{1, 2, 3\}$  and  $n = 1$ , we get 3 equations for our 3 unknown values:

$$\begin{aligned} u_{2,1} + \underbrace{u_{0,1}}_{=1} + \underbrace{u_{1,2}}_{=3} + \underbrace{u_{1,0}}_{=2} - 4u_{1,1} &= 0 \\ u_{3,1} + u_{1,1} + \underbrace{u_{2,2}}_{=3} + \underbrace{u_{2,0}}_{=2} - 4u_{2,1} &= 0 \\ \underbrace{u_{4,1}}_{=4} + u_{2,1} + \underbrace{u_{3,2}}_{=3} + \underbrace{u_{3,0}}_{=2} - 4u_{3,1} &= 0 \end{aligned}$$

In matrix-vector form, these linear equations take the form:

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \\ -9 \end{bmatrix}$$