Final Exam – Practice

MATH 316 — Linear Algebra II
Final Exam: Monday, May 4, 2026

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

- Make sure that you have completed all homework.
- Review the practice problems for both midterms (for the material up to Midterm #2).
- The problems below cover the material since Midterm #2.

Problem 2.

(a) Determine the SVD of $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) Determine the pseudoinverse of A.

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(Hint: We can immediately read off the 0-eigenvector (make sure that's obvious!). It then follows from the spectral theorem that the vector orthogonal to it must be another eigenvector.)

Normalizing, we conclude that $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{50}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

Let's choose $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (the only other choice is $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$). Then, $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

In summary, $A = U \Sigma V^T$ with $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

(b) The pseudoinverse of A is

$$A^+ = V \Sigma^+ U^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & \\ & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Problem 3.

(a) Determine the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

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(b) Determine the best rank 1 approximation of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ has 6-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 4-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A \boldsymbol{v}_2 = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\boldsymbol{u}_3$ needs to be chosen so that the matrix U is orthogonal. To find such a vector, we can start with a random vector like $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and then apply a step of Gram–Schmidt to produce a vector that is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$:

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. We normalize this to $\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Hence, $A = U \Sigma V^T$ with $U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(b) From the SVD we just computed it follows that the best rank 1 approximation of A is (that is, we keep 1 singular value only; together with the first columns of U and V) is

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \sqrt{6} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Comment. Make sure you see that this is the same (but written more efficiently) as $U \Sigma_1 V^T$ where $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 4. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the best rank 1 approximation of A.

(c) Determine the pseudoinverse of A.

(d) Find the smallest solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Then, as a mild check, compare its norm to the obvious solution $\boldsymbol{x} = [\ 1 \ \ 1 \ \ 0\ ]^T$.)

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) = 0 - 1\cdot\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1 \end{bmatrix}\right) + (2-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right)$$
$$= -(2-\lambda) + (2-\lambda)\underbrace{((2-\lambda)(1-\lambda)-1)}_{=\lambda^2-3\lambda+1}$$
$$= (2-\lambda)(\lambda^2-3\lambda) = (2-\lambda)\lambda(\lambda-3).$$

Hence, the eigenvalues are $0, 2, 3$.

- The 0-eigenspace $\operatorname{null}\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
- The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- The 3-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$.

Next, $\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Hence, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

Advanced comment. The computations can be considerably simplified if we start by computing AA^T instead (and then U before V). Can you fill in the details?

(b) From the SVD we just computed it follows that the best rank 1 approximation of A is (that is, we keep 1 singular value only) is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) The pseudoinverse of A is

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}.$$

(d) The smallest solution to $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\boldsymbol{x} = A^+\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 2/3 \\ 1/6 \end{bmatrix}.$$

(For comparison, $\|\boldsymbol{x}\| = \sqrt{11/6} \approx 1.354$ is indeed less than $\|[\ 1\ \ 1\ \ 0\ ]^T\| = \sqrt{2} \approx 1.414$.)

Problem 5.

(a) Precisely state the spectral theorem.

(b) A is singular if and only if $\boxed{\dim \operatorname{null}(A) \qquad\qquad}$.

(c) What exactly does it mean for a matrix A to have full column rank?

(d) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ = \boxed{\qquad\qquad}$.

(e) If A is invertible then its pseudoinverse is $A^+ = \boxed{\qquad}$.

(f) If A has full column rank then its pseudoinverse is $A^+ = \boxed{\qquad}$.

(g) Suppose the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions $\boldsymbol{x}$.

Which of these solutions is produced by $A^+\boldsymbol{b}$? $\boxed{\qquad\qquad\qquad\qquad}$

Solution.

(a) A symmetric (real) matrix A can always be diagonalized. Moreover, all eigenvalues are real and the eigenspaces are orthogonal.

Alternatively: Every symmetric (real) matrix A can be decomposed as $A = PDP^T$, where D is a (real) diagonal matrix and P is orthogonal.

(b) A is singular (i.e. not invertible) if and only if $\dim \operatorname{null}(A) > 0$.

(c) A matrix A has full column rank if its rank equals the number of columns.

(d) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/7 \\ 0 & 0 \end{bmatrix}$.

(e) If A is invertible, then $A^+ = A^{-1}$.

(f) If A has full column rank then its pseudoinverse is $A^+ = (A^T A)^{-1} A^T$.

(g) The one of smallest norm.

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