Homework Set 9

Problem 1

Example 7. Consider the following system of (second-order) initial value problems:

$$y_1'' = 6y_1' - 3y_2' + y_1$$
$$y_2'' = 2y_1' + y_2' - 5y_2$$

Write it as a system of (first-order) differential equations.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 6 & -3 \\ 0 & -5 & 2 & 1 \end{bmatrix} \boldsymbol{y}.$$

Problem 2

Example 8. Without doing any computations, determine e^{Mt} given that

$$M^n = \frac{1}{10} \begin{bmatrix} 4^n + 9 \cdot 2^n & 3 \cdot 4^n - 3 \cdot 2^n \\ 3 \cdot 4^n - 3 \cdot 2^n & 9 \cdot 4^n + 2^n \end{bmatrix}.$$

Solution. We just need to replace 4^n by e^{4t} as well as 2^n by e^{2t} to get

$$e^{Mt} = \frac{1}{10} \begin{bmatrix} e^{4t} + 9e^{2t} & 3e^{4t} - 3e^{2t} \\ 3e^{4t} - 3e^{2t} & 9e^{4t} + e^{2t} \end{bmatrix}.$$

Problem 3

Example 9. Given $e^{Mt} = \begin{bmatrix} -2e^{-t} + 3e^{3t} & -e^{-t} + e^{3t} \\ 6e^{-t} - 6e^{3t} & 3e^{-t} - 2e^{3t} \end{bmatrix}$, determine the matrix M.

Solution. (via matrix powers) Without computations, we can conclude that (by replacing $e^{\lambda t}$ with λ^n)

$$M^n = \begin{bmatrix} -2(-1)^n + 3 \cdot 3^n & -(-1)^n + 3^n \\ 6(-1)^n - 6 \cdot 3^n & 3(-1)^n - 2 \cdot 3^n \end{bmatrix}.$$

In particular, setting $n = 1$, we find $M = \begin{bmatrix} 2 + 3 \cdot 3 & 1 + 3 \\ -6 - 6 \cdot 3 & -3 - 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -24 & -9 \end{bmatrix}$.

Solution. (via derivative) Alternatively, we can find M by computing $\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} = M e^{Mt}$ and then setting $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} = \begin{bmatrix} 2e^{-t} + 9e^{3t} & e^{-t} + 3e^{3t} \\ -6e^{-t} - 18e^{3t} & -3e^{-t} - 6e^{3t} \end{bmatrix}.$$

In particular, now setting $t = 0$, we find $M = \begin{bmatrix} 2 + 9 & 1 + 3 \\ -6 - 18 & -3 - 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -24 & -9 \end{bmatrix}$.

Problem 4

Example 10. How many different Jordan normal forms are there for a 6×6 matrix with eigenvalues $4,4,4,4,9,9$?

Solution. One eigenvalue has multiplicity 4, the other multiplicity 2.

Therefore, there are $5\cdot 2=10$ possible Jordan normal forms.

[See Example 142 in Lecture 26 for where the 5 and 2 come from. Alternatively, you can see this from the listing that follows below.]

Listing all of them.

$$\begin{bmatrix}4&&&&&\\&4&&&&\\&&4&&&\\&&&4&&\\&&&&9&\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&&&&\\&&4&&&\\&&&4&&\\&&&&9&\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&&&&\\&&4&1&&\\&&&4&&\\&&&&9&\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&1&&&\\&&4&&&\\&&&4&&\\&&&&9&\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&1&&&\\&&4&1&&\\&&&4&&\\&&&&9&\\&&&&&9\end{bmatrix},$$

$$\begin{bmatrix}4&&&&&\\&4&&&&\\&&4&&&\\&&&4&&\\&&&&9&1\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&&&&\\&&4&&&\\&&&4&&\\&&&&9&1\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&&&&\\&&4&1&&\\&&&4&&\\&&&&9&1\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&1&&&\\&&4&&&\\&&&4&&\\&&&&9&1\\&&&&&9\end{bmatrix},\begin{bmatrix}4&1&&&&\\&4&1&&&\\&&4&1&&\\&&&4&&\\&&&&9&1\\&&&&&9\end{bmatrix}$$

Problem 5

Example 11. How many different Jordan normal forms are there for a 14×14 matrix with eigenvalues $3,4,4,4,4,5,5,5,5,7,7,7,7,8$?

Solution. The multiplicities of the eigenvalues are $1,4,4,4,1$.

Hence, there are $1\cdot 5\cdot 5\cdot 5\cdot 1=125$ possible Jordan normal forms.

Problem 6

Example 12. Solve the initial value problem $\boldsymbol{y}'=\begin{bmatrix}5&1\\0&5\end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0)=\begin{bmatrix}1\\-2\end{bmatrix}$.

Solution. We can see right away that $A=\begin{bmatrix}5&1\\0&5\end{bmatrix}$ is not diagonalizable (because it is a 2×2 Jordan block).

The solution to the differential equation is

$$\begin{aligned}\boldsymbol{y}(t) &= e^{At}\begin{bmatrix}1\\-2\end{bmatrix}\\ &= e^{5It+Nt}\begin{bmatrix}1\\-2\end{bmatrix} \qquad \text{with } N=\begin{bmatrix}0&1\\0&0\end{bmatrix}\\ &= e^{5It}e^{Nt}\begin{bmatrix}1\\-2\end{bmatrix} \qquad \text{(because } 5It \text{ and } Nt \text{ commute)}\\ &= \begin{bmatrix}e^{5t}&\\&e^{5t}\end{bmatrix}\left(I+Nt+\frac{1}{2}(Nt)^2+\frac{1}{3!}(Nt)^3+\ldots\right)\begin{bmatrix}1\\-2\end{bmatrix}\\ &= \begin{bmatrix}e^{5t}&\\&e^{5t}\end{bmatrix}(I+Nt)\begin{bmatrix}1\\-2\end{bmatrix} \qquad \text{(because } N^2=\mathbf{0}\text{)}\\ &= \begin{bmatrix}e^{5t}&\\&e^{5t}\end{bmatrix}\begin{bmatrix}1&t\\&1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}\\ &= \begin{bmatrix}e^{5t}&\\&e^{5t}\end{bmatrix}\begin{bmatrix}1-2t\\-2\end{bmatrix}=\begin{bmatrix}(1-2t)e^{5t}\\-2e^{5t}\end{bmatrix}.\end{aligned}$$

Problem 7

Example 13. Determine the 2×2 matrix Q for rotation by 49 degrees.

Solution. Recall that the rotation matrix for angle θ is $Q=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Hence, $Q=\begin{bmatrix} \cos(49^\circ) & -\sin(49^\circ) \\ \sin(49^\circ) & \cos(49^\circ) \end{bmatrix}=\begin{bmatrix} \cos\left(49\cdot\frac{2\pi}{360}\right) & -\sin\left(49\cdot\frac{2\pi}{360}\right) \\ \sin\left(49\cdot\frac{2\pi}{360}\right) & \cos\left(49\cdot\frac{2\pi}{360}\right) \end{bmatrix}\approx\begin{bmatrix} 0.6561 & -0.7547 \\ 0.7547 & 0.6561 \end{bmatrix}$.

Problem 8

Example 14. Given $A=\begin{bmatrix} 2-i & -1+i \\ 3-3i & 2+3i \end{bmatrix}$, determine A^*.

Solution. $A^*=(\bar{A})^T=\begin{bmatrix} 2+i & -1-i \\ 3+3i & 2-3i \end{bmatrix}^T=\begin{bmatrix} 2+i & 3+3i \\ -1-i & 2-3i \end{bmatrix}$

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