Homework Set 11

Problem 1

Example 19. Compute the SVD of $A = \begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix}$. That is, decompose A as $A = U\Sigma V^T$.

Solution. (by hand; you will need to show all steps on the final exam)

- First, we need to diagonalize $A^TA = \begin{bmatrix} -3 & -5 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 34 & -12 \\ -12 & 16 \end{bmatrix}$.
  $\det\left(\begin{bmatrix} 34-\lambda & -12 \\ -12 & 16-\lambda \end{bmatrix}\right) = (34-\lambda)(16-\lambda) - 144 = \lambda^2 - 50\lambda + 400 = (\lambda - 10)(\lambda - 40)$
  Hence, the eigenvalues of A^TA are $10, 40$.
  - $\lambda = 10$: $\begin{bmatrix} 24 & -12 \\ -12 & 6 \end{bmatrix} \overset{R_2 + \frac{1}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 24 & -12 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{24}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$
    Hence, the 10-eigenspace has basis $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
  - $\lambda = 40$: $\begin{bmatrix} -6 & -12 \\ -12 & -24 \end{bmatrix} \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -6 & -12 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{6}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$
    Hence, the 40-eigenspace has basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  Thus $A^TA = PDP^T$ with $D = \begin{bmatrix} 40 & \\ & 10 \end{bmatrix}$ and $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$.
  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA = V\Sigma^2V^T$, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{40} & \\ & \sqrt{10} \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{40}}\begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{200}}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{50}}\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{40} & \\ & \sqrt{10} \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution. (using Sage) We obtain the same solution (up to a sign in U and V):

```
>>> A = matrix(RDF, [[-3,4],[-5,0]])
>>> U,S,V = A.SVD()
>>> U
```

$$\begin{pmatrix} -0.7071067811865475 & -0.7071067811865476 \\ -0.7071067811865476 & 0.7071067811865475 \end{pmatrix}$$

```
>>> S
```

$$\begin{pmatrix} 6.324555320336758 & 0.0 \\ 0.0 & 3.16227766016838 \end{pmatrix}$$

```
>>> V
```

$$\begin{pmatrix} 0.8944271909999159 & -0.44721359549995804 \\ -0.44721359549995804 & -0.8944271909999159 \end{pmatrix}$$

Problem 2

Example 20. Compute the SVD of $A = \begin{bmatrix} -6 & 2 \\ 6 & -2 \end{bmatrix}$.

Solution. (by hand; you will need to show all steps on the final exam)

- First, we need to diagonalize $A^T A = \begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -6 & 2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 72 & -24 \\ -24 & 8 \end{bmatrix}$.
  $\det\left(\begin{bmatrix} 72-\lambda & -24 \\ -24 & 8-\lambda \end{bmatrix}\right) = (72-\lambda)(8-\lambda) - 576 = \lambda^2 - 80\lambda = \lambda(\lambda - 80)$
  Hence, the eigenvalues of $A^T A$ are $0, 80$.
  - $\lambda = 0$: $\begin{bmatrix} 72 & -24 \\ -24 & 8 \end{bmatrix} \overset{R_2 + \frac{1}{3}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 72 & -24 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{72}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$
    Hence, the 0-eigenspace has basis $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
  - $\lambda = 80$: $\begin{bmatrix} -8 & -24 \\ -24 & -72 \end{bmatrix} \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -8 & -24 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{8}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$
    Hence, the 80-eigenspace has basis $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

  Thus $A^T A = PDP^T$ with $D = \begin{bmatrix} 80 & \\ & 0 \end{bmatrix}$ and $P = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.
  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^T A = V\Sigma^2 V^T$, we conclude that $V = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{80} & \\ & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{80}}\begin{bmatrix} -6 & 2 \\ 6 & -2 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{800}}\begin{bmatrix} 20 \\ -20 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
  We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.
  For instance, $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so that $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{80} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution. (using Sage) We obtain the same solution (up to a sign in U and V):

```
>>> A = matrix(RDF, [[-6,2],[6,-2]])
>>> U,S,V = A.SVD()
>>> U
```

$$\begin{pmatrix} -0.7071067811865472 & 0.7071067811865472 \\ 0.7071067811865472 & 0.7071067811865475 \end{pmatrix}$$

```
>>> S
```

$$\begin{pmatrix} 8.944271909999161 & 0.0 \\ 0.0 & 2.1065000811460205 \times 10^{-16} \end{pmatrix}$$

```
>>> V
```

$$\begin{pmatrix} 0.9486832980505138 & -0.31622776601683783 \\ -0.31622776601683783 & -0.9486832980505138 \end{pmatrix}$$

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Problem 3

Example 21. Compute the SVD of $A = \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix}$.

Solution. **(by hand; you will need to show all steps on the final exam)**

- First, we need to diagonalize $A^T A = \begin{bmatrix} -7 & 5 & 1 \\ -1 & -5 & 3 \end{bmatrix} \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 75 & -15 \\ -15 & 35 \end{bmatrix}$.

  $\det\left(\begin{bmatrix} 75-\lambda & -15 \\ -15 & 35-\lambda \end{bmatrix} \right) = (75-\lambda)(35-\lambda) - 225 = \lambda^2 - 110\lambda + 2400 = (\lambda-30)(\lambda-80)$

  Hence, the eigenvalues of $A^T A$ are $30, 80$.

  - $\lambda = 30$: $\begin{bmatrix} 45 & -15 \\ -15 & 5 \end{bmatrix} \overset{R_2 + \frac{1}{3}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 45 & -15 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{45}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$

    Hence, the 30-eigenspace has basis $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

  - $\lambda = 80$: $\begin{bmatrix} -5 & -15 \\ -15 & -45 \end{bmatrix} \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -5 & -15 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{5}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

    Hence, the 80-eigenspace has basis $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

  Thus $A^T A = PDP^T$ with $D = \begin{bmatrix} 80 & \\ & 30 \end{bmatrix}$ and $P = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^T A = V \Sigma^2 V^T$, we conclude that $V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{30} \\ 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{80}} \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{800}} \begin{bmatrix} 20 \\ -20 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

  Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A \boldsymbol{v}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{300}} \begin{bmatrix} -10 \\ -10 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

  We cannot obtain $\boldsymbol{u}_3$ like this because there is no σ_3. We need to choose $\boldsymbol{u}_3$ so that U is orthogonal.

  To find a vector that is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we compute:

  $\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \overset{R_2 + R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \overset{-\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \overset{R_1 + R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$

  Therefore, $\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

  Normalizing $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ to $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, we conclude that $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{30} \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution. (using Sage) We obtain the same solution (up to a sign in U and V):

```
>>> A = matrix(RDF, [[-7,-1],[5,-5],[1,3]])
>>> U,S,V = A.SVD()
>>> U
```

$$\begin{pmatrix} -0.7071067811865476 & -0.577350269189626 & 0.40824829046386296 \\ 0.7071067811865477 & -0.5773502691896258 & 0.4082482904638629 \\ -1.4525337733367862\times 10^{-17} & 0.5773502691896257 & 0.816496580927726 \end{pmatrix}$$

```
>>> S
```

$$\begin{pmatrix} 8.94427190999157 & 0.0 \\ 0.0 & 5.477225575051662 \\ 0.0 & 0.0 \end{pmatrix}$$

```
>>> V
```

$$\begin{pmatrix} 0.9486832980505138 & 0.316227766016838 \\ -0.316227766016838 & 0.9486832980505138 \end{pmatrix}$$

Problem 4

Example 22. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$ (without computing the SVD first).

Solution. This matrix clearly has full column rank (because the two columns are not multiples of each other). Hence, $A^+ = (A^TA)^{-1}A^T = \begin{bmatrix} 13 & -6 \\ -6 & 13 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133}\begin{bmatrix} 13 & 6 \\ 6 & 13 \end{bmatrix}\begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133}\begin{bmatrix} 8 & 12 & 39 \\ -27 & 26 & 18 \end{bmatrix}$.

Problem 5

Example 23. Determine the pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix}$.

Solution. For such diagonal matrices, we only need to invert the diagonal entries and transpose the dimensions. $A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/5 \\ 0 & 0 \end{bmatrix}$

Problem 6

Example 24. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$ (by computing the SVD first).

Solution. (skipping most work) As observed in Examples 181 and 182 in the lecture notes, we can avoid almost all computations and conclude that, if $A = \boldsymbol{a}^T$ is a row vector, then

$$A^+ = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2} = \frac{1}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Solution. (too much work but good pratice) Let us embrace the opportunity to practice. We first compute the SVD of A:

First, we diagonalize $A^TA = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} [\, 2 \;\; -2 \;\; 1 \,] = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}$. Let us write $|A|$ for $\det(A)$:

$$\begin{vmatrix} 4-\lambda & -4 & 2 \\ -4 & 4-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = (4-\lambda)\cdot \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - (-4)\cdot \begin{vmatrix} -4 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2\cdot \begin{vmatrix} -4 & 4-\lambda \\ 2 & -2 \end{vmatrix}$$
$$= (4-\lambda)\cdot(\lambda^2 - 5\lambda) + 4\cdot(4\lambda) + 2\cdot(2\lambda) = -\lambda^3 + 9\lambda^2 = \lambda^2(9-\lambda)$$

Hence, the eigenvalues of A^TA are $9, 0, 0$.

- $\lambda = 9$: $\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \overset{\substack{R_2 - \frac{4}{5}R_1 \Rightarrow R_2 \\ R_3 + \frac{2}{5}R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & -\frac{18}{5} & -\frac{36}{5} \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix}$
  $\overset{\substack{-\frac{1}{5}R_1 \Rightarrow R_1 \\ -\frac{5}{9}R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - \frac{4}{5}R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, the 9-eigenspace has basis $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

- $\lambda = 0$: $\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \overset{\substack{R_2 + R_1 \Rightarrow R_2 \\ R_3 - \frac{1}{2}R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{4}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

  Hence, the 0-eigenspace has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$. For the SVD we have to turn this basis into an orthogonal one.

  Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

Thus $A^TA = PDP^T$ with $D = \begin{bmatrix} 9 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

[We had to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA = V\Sigma^2V^T$, we conclude that $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$ and $\Sigma = [\, 3 \;\; 0 \;\; 0 \,]$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{3} [\, 2 \;\; -2 \;\; 1 \,] \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = 1$.

Hence, $A = U\Sigma V^T$ with $U = [\, 1 \,]$, $\Sigma = [\, 3 \;\; 0 \;\; 0 \,]$, $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

Using the SVD of A, we can easily obtain its pseudoinverse:

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} [\, 1 \,] = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Comments. This was good practice computing SVDs but we did way too much work: Can you adjust our approach from working with A^TA (3×3) to working with AA^T (1×1)? On the other hand, can you see why it was clear that A^TA was going to have 0 as a repeated eigenvalue? Can you see why the last two columns of P are irrelevant in our work? Can you see how we could have obtained the first column of P without computation?

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Problem 7

Example 25. Find the smallest norm solution to $4x_1+3x_2+5x_3=3$.

Solution. If $A=\begin{bmatrix} 4 & 3 & 5 \end{bmatrix}$, then the smallest norm solution is $\boldsymbol{x}=A^+\begin{bmatrix} 3 \end{bmatrix}$.

From earlier computations (see, for instance, Example 177) we know that $A^+=\frac{1}{4^2+3^2+5^2}\begin{bmatrix} 4\\3\\5 \end{bmatrix}=\frac{1}{50}\begin{bmatrix} 4\\3\\5 \end{bmatrix}$.

Hence, the smallest norm solution is $\boldsymbol{x}=A^+\begin{bmatrix} 3 \end{bmatrix}=\frac{3}{50}\begin{bmatrix} 4\\3\\5 \end{bmatrix}$.

Problem 8

Example 26. Determine the best rank 1 approximation of $A=\begin{bmatrix} 1 & -2\\ 0 & -1\\ 1 & 0 \end{bmatrix}$.

Solution. We first compute the SVD of A:

- First, we need to diagonalize $A^TA=\begin{bmatrix} 1 & 0 & 1\\ -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2\\ 0 & -1\\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 2 & -2\\ -2 & 5 \end{bmatrix}$.

  $\det\left(\begin{bmatrix} 2-\lambda & -2\\ -2 & 5-\lambda \end{bmatrix}\right)=(2-\lambda)(5-\lambda)-4=\lambda^2-7\lambda+6=(\lambda-1)(\lambda-6)$

  Hence, the eigenvalues of A^TA are $6,1$.

  - $\lambda=6$: $\begin{bmatrix} -4 & -2\\ -2 & -1 \end{bmatrix}\overset{R_2-\frac{1}{2}R_1\Rightarrow R_2}{\rightsquigarrow}\begin{bmatrix} -4 & -2\\ 0 & 0 \end{bmatrix}\overset{-\frac{1}{4}R_1\Rightarrow R_1}{\rightsquigarrow}\begin{bmatrix} 1 & \frac{1}{2}\\ 0 & 0 \end{bmatrix}$

    Hence, the 6-eigenspace has basis $\begin{bmatrix} -1/2\\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} -1\\ 2 \end{bmatrix}$.

  - $\lambda=1$: $\begin{bmatrix} 1 & -2\\ -2 & 4 \end{bmatrix}\overset{R_2+2R_1\Rightarrow R_2}{\rightsquigarrow}\begin{bmatrix} 1 & -2\\ 0 & 0 \end{bmatrix}$

    Hence, the 1-eigenspace has basis $\begin{bmatrix} 2\\ 1 \end{bmatrix}$.

  Thus $A^TA=PDP^T$ with $D=\begin{bmatrix} 6 & \\ & 1 \end{bmatrix}$ and $P=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2\\ 2 & 1 \end{bmatrix}$.

  [We had to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA=V\Sigma^2V^T$, we conclude that $V=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2\\ 2 & 1 \end{bmatrix}$ and $\Sigma=\begin{bmatrix} \sqrt{6} & 0\\ 0 & 1\\ 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i=\sigma_i\boldsymbol{u}_i$, we find $\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2\\ 0 & -1\\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1\\ 2 \end{bmatrix}=\frac{1}{\sqrt{30}}\begin{bmatrix} -5\\ -2\\ -1 \end{bmatrix}$.

  For the rank 1 approximation, we only need the first column of U, so we stop here.

Hence, $A=U\Sigma V^T$ with $U=\begin{bmatrix} -5/\sqrt{30} & * & *\\ -2/\sqrt{30} & * & *\\ -1/\sqrt{30} & * & * \end{bmatrix}$, $\Sigma=\begin{bmatrix} \sqrt{6} & 0\\ 0 & 1\\ 0 & 0 \end{bmatrix}$, $V=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2\\ 2 & 1 \end{bmatrix}$.

From the SVD of A, we obtain the best rank 1 approximation by only using the first columns of U and V (and truncating Σ to a 1×1 matrix):

Thus, the best rank 1 approximation of A is $\frac{1}{\sqrt{30}}\begin{bmatrix} -5\\ -2\\ -1 \end{bmatrix}\begin{bmatrix} \sqrt{6} \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1\\ 2 \end{bmatrix}^T=\sqrt{\frac{6}{30\cdot 5}}\begin{bmatrix} -5\\ -2\\ -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} 5 & -10\\ 2 & -4\\ 1 & -2 \end{bmatrix}$.

Comment. Like for U, we could have omitted the computation of the 1-eigenvector (second column of V).

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