Application: Linear differential equations

Example 120. (warmup) Solve the **differential equation** (DE) $y' = 2$.

Solution. From calculus, we know that the solutions are of the form $y(t) = 2t + C$.

Comment. To get a unique solution, we need to specify additional information, like an initial condition.

Example 121. (warmup) Solve the **initial value problem** (IVP) $y' = 2$, $y(0) = 1$.

Solution. This has the unique solution $y(t) = 2t + 1$.

Example 122. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Solution. $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

(exponential function) e^t is the unique solution to $y' = y$, $y(0) = 1$.

From here, it follows that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$.

The latter is the Taylor series for e^t at $t = 0$ that we have seen in Calculus II.

Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{t^3}{3!} = \frac{t^2}{2!}$.

Example 123. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Solution. Indeed, if $y(t) = Ce^{3t}$, then $y'(t) = 3Ce^{3t} = 3y(t)$.

Comment. It is important to realize that we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

Example 124. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Solution. As in the previous example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$.
Since $y(0) = Ce^0 = C = y_0$, we conclude that the unique solution to the IVP is $y(t) = e^{at}y_0$.

Comment. It looks silly to write $e^{at}y_0$ instead of y_0e^{at} here, but we will soon replace the number a with a matrix A, and in that case only $e^{At}y_0$ makes sense.

Example 125. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrrcrcl} y_1' &=& 2y_1 & & & \quad & y_1(0) &=& 1 \\ y_2' &=& -y_1 & +3y_2 & +y_3 & & y_2(0) &=& 0 \\ y_3' &=& -y_1 & +y_2 & +3y_3 & & y_3(0) &=& 2 \end{array}$$

In matrix form, this becomes

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The key idea will be to solve $\boldsymbol{y}' = A\boldsymbol{y}$ by introducing e^{At}.

Theorem 126. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Armin Straub
straub@southalabama.edu

Recall from Example 124 that the solution to $y' = ay$, $y(0) = y_0$ is $y(t) = e^{at} y_0$. Here, however, At is a matrix and so we need to make sense of the matrix exponential. Next time, we will define e^A by the familiar Taylor series for e^x.

Definition 127. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots$$

Why? As a consequence of this definition (which is the motivation for that definition in the first place),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e^{At} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots\right] \\ &= 0 + A + A^2 t + \frac{1}{2!} A^3 t^2 + \cdots = A e^{At}. \end{aligned}$$

Therefore, $\boldsymbol{y}(t) = e^{At} \boldsymbol{y}_0$ indeed solves the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

How to actually compute e^A? Well, this Taylor series involves the powers A^n of A. How would you compute, say, A^{100}? The answer is diagonalization!

Theorem 128. Suppose $A = PDP^{-1}$. Then, $e^A = P e^D P^{-1}$.

Why? Recall that, if $A = PDP^{-1}$, then $A^n = PD^n P^{-1}$.

$$\begin{aligned} e^A &= I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots \\ &= I + PDP^{-1} + \frac{1}{2!} PD^2 P^{-1} + \frac{1}{3!} PD^3 P^{-1} + \cdots \\ &= P\left(I + D + \frac{1}{2!} D^2 + \frac{1}{3!} D^3 + \cdots\right) P^{-1} = P e^D P^{-1} \end{aligned}$$

Comment. By the same argument, if $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ for every "nice" function f.
Here, "nice" means that f has a convergent Taylor series $f(x) = \sum_{n \geqslant 0} a_n x^n$.
More explicitly, if $A = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,P^{-1}$, then $f(A) = P\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))\,P^{-1}$.

Example 129. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

Example 130. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $At = \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix}$, $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.