Example 135. Consider the following system of (second-order) initial value problems:

$$\begin{array}{l} y_1'' = 2y_1' - 3y_2' + 7y_2 \\ y_2'' = 4y_1' + y_2' - 5y_1 \end{array} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Example 136. Suppose that $e^{Mt} = \frac{1}{10}\begin{bmatrix} e^t + 9e^{2t} & 3e^t - 3e^{2t} \\ 3e^t - 3e^{2t} & 9e^t + e^{2t} \end{bmatrix}$.

(a) Without doing any computations, determine M^n.

(b) What is M?

(c) Without doing any computations, determine the eigenvalues and eigenvectors of M.

Solution.

(a) Recall that $e^{Mt} = Pe^{Dt}P^{-1}$ while $M^n = PD^nP^{-1}$, provided that $M = PDP^{-1}$. The fact the formula for e^{Mt} features e^t and e^{2t}, means that the eigenvalues of M must be 1 and 2. Hence,

$$D = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix}, \quad D^n = \begin{bmatrix} 1 & \\ & 2^n \end{bmatrix}.$$

Therefore, we just need to replace e^t by $1^n = 1$ as well as e^{2t} by 2^n to get:

$$M^n = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2^n & 3 - 3 \cdot 2^n \\ 3 - 3 \cdot 2^n & 9 + 2^n \end{bmatrix}$$

(b) In particular, we see that the underlying matrix is $M = M^1 = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2 & 3 - 3 \cdot 2 \\ 3 - 3 \cdot 2 & 9 + 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$.

[Alternatively, we can find M by computing $\frac{\mathrm{d}}{\mathrm{d}t}e^{Mt} = Me^{Mt}$ and then setting $t = 0$.]

(c) The eigenvalues are 1 and 2.

Looking at the coefficients of e^t in the first column of e^{Mt}, we can see that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a 1-eigenvector.

[We can also look the second column of e^{Mt}, to obtain $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ which is a multiple and thus equivalent.]

Likewise, we find that $\begin{bmatrix} 9 \\ -3 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a 2-eigenvector.

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The Jordan normal form

Note that we currently only know how to compute e^{At} when A is diagonalizable. Our next goal is to see how one can compute the matrix exponential for all matrices.

Example 137. Diagonalize, if possible, the matrix $A = \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}$.

Solution. The eigenvalues of A are $4, 4$.

However, the 4-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1-dimensional.

Hence, A is not diagonalizable.

Definition 138. A **$\boldsymbol{\lambda}$-Jordan block** is a matrix of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$.

Note that if this matrix is $m \times m$, then its only eigenvalue is λ (repeated m times).

As in the previous example, the λ-eigenspace is 1-dimensional (which is as small as possible).

Theorem 139. (Jordan normal form) Every $n \times n$ matrix A can be written as $A = PJP^{-1}$, where J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}$$

with each J_i a Jordan block. J is called the **Jordan normal form** of A.

Up to the ordering of the Jordan blocks, the Jordan normal form of A is unique.

Comment. If A is diagonalizable, then J is just a usual diagonal matrix.