

Review. Jordan normal form

Example 140. What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues 4, 4, 4?

Solution. $\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$

The dimension of the 4-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you know from diagonalization).

Example 141.

- What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues 3, 3, 3?
- What are the possible Jordan normal forms of a 4×4 matrix with eigenvalues 3, 3, 3, 3?
- What if the matrix is 5×5 and has eigenvalues 4, 4, 3, 3, 3?

Solution.

(a) $\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 3 & 1 & \\ & 3 & \\ & & 3 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you known from diagonalization).

(b) Now, there are 5 possibilities:

$$\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 4, 3, 2, 2, 1, respectively.

(c) $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$

Note that this is just all possible (namely, 3) Jordan normal forms of a 3×3 matrix with eigenvalues 3, 3, 3 combined with all possible (namely, 2) Jordan normal forms of a 2×2 matrix with eigenvalues 4, 4. In total, that makes $3 \cdot 2 = 6$ possibilities.

Comment. Let $p(n)$ be the number of inequivalent Jordan normal forms of an $n \times n$ matrix with a single eigenvalue, n times repeated. We have seen that $p(2) = 2$, $p(3) = 3$, $p(4) = 5$. Note that $p(n)$ is equal to the number of ways of writing n as an ordered sum of positive integers: for instance, $p(4) = 5$ because $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$.

$p(n)$ is referred to as the **partition function** and, surprisingly, is a remarkably interesting mathematical object.

[https://en.wikipedia.org/wiki/Partition_function_\(number_theory\)](https://en.wikipedia.org/wiki/Partition_function_(number_theory))

Example 142. (summary of small cases)

(a) There are 2 possible Jordan normal forms of a 2×2 matrix with eigenvalues λ, λ .

Namely. $\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$

(b) There are 3 possible Jordan normal forms of a 3×3 matrix with eigenvalues $\lambda, \lambda, \lambda$.

Namely. $\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$

(c) There are 5 possible Jordan normal forms of a 4×4 matrix with eigenvalues $\lambda, \lambda, \lambda, \lambda$.

Namely. $\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & 1 & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$

Example 143. What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues 3, 3, 7, 7, 7, 7?

Solution. There are $2 \cdot 5 = 10$ possible Jordan normal forms for such a matrix:

$$\begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & 1 & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & 1 & \\ & & & 7 & 1 & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & 1 & 1 \\ & & & 7 & 1 & 1 \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & 1 & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & 1 & \\ & & & 7 & 1 & 1 \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & 1 & 1 \\ & & & 7 & 1 & 1 \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}$$

Example 144. How many different Jordan normal forms are there in the following cases?

- (a) A 8×8 matrix with eigenvalues 1, 1, 2, 2, 2, 4, 4, 4?
- (b) A 11×11 matrix with eigenvalues 1, 1, 1, 2, 2, 2, 2, 4, 4, 4, 4?

Solution.

- (a) $2 \cdot 3 \cdot 3 = 18$ possible Jordan normal forms
- (b) $3 \cdot 5 \cdot 5 = 75$ possible Jordan normal forms

The matrix exponential of matrices that are not diagonalizable

Review.

- Let A be $n \times n$. The matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

Then, $\frac{d}{dt}e^{At} = Ae^{At}$.

Why? $\frac{d}{dt}e^{At} = \frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \dots = Ae^{At}$

- If $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.
- The solution to $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$ is $\mathbf{y}(t) = e^{At}\mathbf{y}_0$.
Why? Because $\mathbf{y}'(t) = Ae^{At}\mathbf{y}_0 = A\mathbf{y}(t)$ and $\mathbf{y}(0) = e^{0A}\mathbf{y}_0 = \mathbf{y}_0$.

Example 145. The matrix exponential shares many other properties of the usual exponential:

- $e^A e^B = e^{A+B} = e^B e^A$ if $AB = BA$

Why the condition $AB = BA$? By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \dots$. In order to simplify that to

$$e^A e^B = \left(I + A + \frac{A^2}{2!} + \dots \right) \left(I + B + \frac{B^2}{2!} + \dots \right),$$

we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- e^A is invertible and $(e^A)^{-1} = e^{-A}$

Why? That actually follows from the previous property.

Example 146. Compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Solution. Note that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{At} = I + At + \frac{t^2}{2!} A^2 + \dots = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

Example 147. Compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution. Note that $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, $e^{At} = I + At + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \dots = I + At + \frac{1}{2} A^2 t^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix}$.

Example 148. Compute e^{At} for $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$.

Solution.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $2I$ and N commute.

Hence, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!} N^2 + \dots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & t e^{2t} \\ & e^{2t} \end{bmatrix}$.

Advanced. Can you show that, similarly, $A^n = \begin{bmatrix} 2^n & n 2^{n-1} \\ & 2^n \end{bmatrix}$?

Example 149. Solve the differential equation

$$\mathbf{y}' = \underbrace{\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}}_A \mathbf{y}, \quad \mathbf{y}(0) = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{y}_0}$$

Solution. Repeating the work in the previous example, the solution to the differential equation is

$$\begin{aligned} \mathbf{y}(t) &= e^{At} \mathbf{y}_0 \\ &= e^{2It + Nt} \mathbf{y}_0 \quad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= e^{2It} e^{Nt} \mathbf{y}_0 \quad (\text{because } 2It \text{ and } Nt \text{ commute}) \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \left(I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \dots \right) \mathbf{y}_0 \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} (I + Nt) \mathbf{y}_0 \quad (\text{because } N^2 = \mathbf{0}) \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix}. \end{aligned}$$

Check. We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$. Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

Comment. For applications, having solutions like $te^{\lambda t}$ or $t \cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.

Important comment. Note that we can immediately see from the solution that the original matrix A is not diagonalizable: there is a term te^{2t} , whereas in the diagonalizable case we would only see exponentials like e^{2t} by themselves.

In our upcoming discussion of complex numbers we will see that e^{2it} (here, $2i$ would be the eigenvalue) can be rewritten in terms of $\cos(2t)$ and $\sin(2t)$. Both of these are periodic and bounded, so that the same is true for every linear combination.

In that case, if the eigenvalue $2i$ was repeated in such a way that the matrix A is not diagonalizable, then we would get the functions $t \cos(2t)$ and $t \sin(2t)$ in our solutions. These, however, are not bounded! This phenomenon (getting solutions that are unbounded under the right/wrong circumstances) is called **resonance**.

<https://en.wikipedia.org/wiki/Resonance>

Understanding when resonance occurs is of crucial importance for practical applications.