

**Review.** Jordan normal form

### Rotation matrices

**Example 150.** Write down a  $2 \times 2$  matrix  $Q$  for rotation by angle  $\theta$  in the plane.

**Comment.** Why should we even be able to represent something like rotation by a matrix? Meaning that  $Q\mathbf{x}$  should be the vector  $\mathbf{x}$  rotated by  $\theta$ . Recall from Linear Algebra I that every **linear map** can be represented by a matrix. Then think about why rotation is a linear map.

**Solution.** We can determine  $Q$  by figuring out  $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  (the first column of  $Q$ ) and  $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  (the second column of  $Q$ ).

Since  $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$  and  $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ , we conclude that  $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ .

**Comment.** Note that we don't need previous knowledge of  $\cos$  and  $\sin$ . We could have introduced these trig functions on the spot.

**Comment.** Note that it is geometrically obvious that  $Q$  is orthogonal. (Why?)

It is clear that  $\left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|^2 = 1$ . Noting that  $\left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|^2 = \cos^2\theta + \sin^2\theta$ , we have rediscovered Pythagoras.

**Advanced comment.** Actually, every orthogonal  $2 \times 2$  matrix  $Q$  with  $\det(Q) = 1$  is a rotation by some angle  $\theta$ . Orthogonal matrices with  $\det(Q) = -1$  are reflections.

**Example 151.** As in the previous example, let  $Q_\theta$  be the  $2 \times 2$  matrix for rotation by angle  $\theta$  in the plane. What is  $Q_\alpha Q_\beta$ ?

**Solution.** Note that  $Q_\alpha Q_\beta \mathbf{x}$  first rotates  $\mathbf{x}$  by angle  $\beta$  and then by angle  $\alpha$ . For geometric reasons, it is obvious that this is the same as if we rotated  $\mathbf{x}$  by  $\alpha + \beta$ . It follows that  $Q_\alpha Q_\beta = Q_{\alpha+\beta}$ .

**Comment.** This allows us to derive interesting trig identities:

$$Q_\alpha Q_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \dots \\ \dots & \dots \end{bmatrix}$$

$$Q_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

It follows that  $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ .

**Comment.** If we set  $\beta = \alpha$ , this simplifies to  $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$ , the double angle formula that you have probably used countless times in Calculus.

**Comment.** Similarly, we find an identity for  $\sin(\alpha + \beta)$ . Spell it out!

### More on complex numbers

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as  $z = x + iy$  with real  $x, y$ .

- Here, the imaginary unit  $i$  is characterized by solving  $x^2 = -1$ .

**Important observation.** The same equation is solved by  $-i$ . This means that, algebraically, we cannot distinguish between  $+i$  and  $-i$ .

- The **conjugate** of  $z = x + iy$  is  $\bar{z} = x - iy$ .

**Important comment.** Since we cannot algebraically distinguish between  $\pm i$ , we also cannot distinguish between  $z$  and  $\bar{z}$ . That's the reason why, in problems involving only real numbers, if a complex number  $z = x + iy$  shows up, then its **conjugate**  $\bar{z} = x - iy$  has to show up in the same manner. With that in mind, have another look at Example 89.

- The **absolute value** of the complex number  $z = x + iy$  is  $|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$ .
- The **norm** of the complex vector  $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$  is  $\|\mathbf{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$ .  
Note that  $\|\mathbf{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\mathbf{z}}^T \mathbf{z}$ .

**Definition 152.**

- For every matrix  $A$ , its **conjugate transpose** is  $A^* = (\bar{A})^T$ .
- The **dot product** (inner product) of complex vectors is  $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^* \mathbf{w}$ .
- A complex  $n \times n$  matrix  $A$  is **unitary** if  $A^* A = I$ .

**Comment.**  $A^*$  is also written  $A^H$  (or  $A^\dagger$  in quantum mechanics) and called the Hermitian conjugate.

**Comment.** For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

**Comment.** Unitary matrices are the complex version of orthogonal matrices. (A real matrix is unitary if and only if it is orthogonal.)

**Example 153.** Determine  $A^*$  if  $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$ .

**Solution.**  $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

**Example 154.** What is  $\frac{1}{2+3i}$ ?

**Solution.**  $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$ .

**In general.**  $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$