Review. Complex numbers

Example 155. (April Fools' Day!) Foul play with complex numbers:

$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i i = -1.$

When using the principal square-root (which basically takes the positive root, that is, the one with positive real part), the rule $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ does not hold universally (so the trouble lies with the third equality). It does hold if, for instance, $a \geqslant 0$ or $b \geqslant 0$. Apparently, this trouble historically resulted in controversy around complex numbers, with some mathematicians rejecting them outright.

Example 156. (April Fools' Day!) What is the norm of the vector $\begin{bmatrix} 1 \\ i \end{bmatrix}$? Could it be zero?!

Solution. $\left\| \begin{bmatrix} 1 \\ i \end{bmatrix} \right\| = \sqrt{|1|^2 + |i|^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

Comment. If we carelessly use $\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$ (which works fine only when a and b are real numbers), then we would (incorrectly) get that the vector has norm 0.

Example 157. What is the norm of the vector $\begin{bmatrix} 4 - i \\ 2 + 3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 4 - i \\ 2 + 3i \end{bmatrix} \right\| = \sqrt{|4 - i|^2 + |2 + 3i|^2} = \sqrt{(4^2 + (-1)^2) + (2^2 + 3^2)} = \sqrt{30}$

$\left\| \begin{bmatrix} 1 - i \\ 2 + 3i \end{bmatrix} \right\|^2 = \begin{bmatrix} 1 + i & 2 - 3i \end{bmatrix} \begin{bmatrix} 1 - i \\ 2 + 3i \end{bmatrix} = |1 - i|^2 + |2 + 3i|^2 = 2 + 13$. Hence, $\left\| \begin{bmatrix} 1 - i \\ 2 + 3i \end{bmatrix} \right\| = \sqrt{15}$.

Example 158. True or false? Every $n \times n$ matrix A always has an eigenvector v.

Solution. True! Every $n \times n$ matrix always has exactly n eigenvalues (if we allow complex eigenvalues and count with repetition).

If λ is one of those eigenvalues, then the dimension of the λ-eigenspace is at least 1 (because $\det(A - \lambda I) = 0$ so that $A\boldsymbol{v} = \lambda \boldsymbol{v}$ has nonzero solutions $\boldsymbol{v}$).

Example 159. (April Fools' Day!) Let A be an $n \times n$ matrix. By the previous example, we can find an eigenvector $\boldsymbol{v}$ and an eigenvalue λ such that $A\boldsymbol{v} = \lambda \boldsymbol{v}$.

Let us rewrite that as $A\boldsymbol{v} = \lambda I \boldsymbol{v}$ where I is the $n \times n$ identity matrix. Does it follow that $A = \lambda I$?

Solution. No, that is (of course) not a valid conclusion. One cannot cancel a vector $\boldsymbol{v}$ in this way.

Recall, more generally for matrices, that in order to conclude from $AB = CB$ that $A = C$ we need extra information on B such as B being invertible.

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