Notes for Lecture 1

Mon, 1/12/2026

Review: Matrix calculus

Example 1. Matrix multiplication is not commutative!

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 10 \end{bmatrix}$

  Multiplication (on the right) with that "almost identity matrix" is performing the column operation $C_2 + 2C_1 \Rightarrow C_2$ (i.e. 2 times the first column is added to the second column).

- $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix}$

  Multiplication (on the left) with the same matrix is performing the row operation $R_1 + 2R_2 \Rightarrow R_1$.

  First comment. This indicates a second interpretation of matrix multiplication: instead of taking linear combinations of columns of the first matrix, we can also take linear combinations of rows of the second matrix.

  Second comment. The row operations we are doing during Gaussian elimination can be realized by multiplying (on the left) with "almost identity matrices".

Example 2. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

If you know about the dot product, do you see a connection with the first case?

Example 3. Suppose A is $m \times n$ and B is $p \times q$. When does AB make sense? In that case, what are the dimensions of AB?

AB makes sense if $n = p$. In that case, AB is a $m \times q$ matrix.

Example 4. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Hence, the two matrices on the left are inverses of each other: $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

Example 5. The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{provided that } ad - bc \neq 0$$

Let's check that! $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = I_2$

In particular, a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad - bc \neq 0$.

Recall that this is the **determinant**: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

In particular:

$$\det(A) = 0 \quad \iff \quad A \text{ is not invertible}$$

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Similarly, for $n\times n$ matrices A:

	A is invertible	(i.e. there is a matrix A^{-1} such that $AA^{-1}=I$)
$\Longleftrightarrow$	$\det(A)\neq 0$	
$\Longleftrightarrow$	$A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution	(namely, $\boldsymbol{x}=A^{-1}\boldsymbol{b}$)

Comment. Why is it not common to write $\frac{1}{A}$ instead of A^{-1}?

The notation $\frac{1}{A}$ easily leads to ambiguities: for instance, should $\frac{B}{A}$ mean BA^{-1} or should it mean $A^{-1}B$?

[Of course, one could try to avoid this by notations like B/A which would more clearly mean BA^{-1}. It's just not common and doesn't have any real advantages.]

Example 6.

$$\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ -4 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} -7 & 2 & 3\\ -16 & 5 & 6\\ -25 & 8 & 9 \end{bmatrix}$$

Multiplication (on the right) with that "almost identity matrix" is performing the column operation $C_1-4C_2\Rightarrow C_1$ (i.e. -4 times the second column is added to the first column).

$$\begin{bmatrix} 1 & 0 & 0\\ -4 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9 \end{bmatrix}=\begin{bmatrix} 1 & 2 & 3\\ 0 & -3 & -6\\ 7 & 8 & 9 \end{bmatrix}$$

Multiplication (on the left) with the same matrix is performing the row operation $R_2-4R_1\Rightarrow R_2$.

Comment (again). The row operations we are doing during Gaussian elimination can all be realized by multiplying (on the left) with "almost identity matrices".

These matrices are called **elementary matrices** (they are obtained by performing a single elementary row operation on an identity matrix).

Elementary matrices are **invertible** because elementary row operations are reversible:

$$\begin{bmatrix} 1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix},\quad \begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

Example 7. Let us do Gaussian elimination on $A=\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ until we have an echelon form:

$$A=\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \overset{R_2-2R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

As last class, the row operation can be encoded by multiplication with an "almost identity matrix" E:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}}_{A}=\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}}_{U}$$

Since $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (no calculation needed; this is the row operation $R_2+2R_1\Rightarrow R_2$ which reverses our above operation), this means that

$$A=E^{-1}U=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}.$$

We factored A as the product of a lower and an upper triangular matrix!

> $A=LU$ is known as the **LU decomposition** of A.
>
> L is lower triangular, U is upper triangular.

If A is $m\times n$, then L is an invertible lower triangular $m\times m$ matrix, and U is a usual **echelon form** of A. Every matrix A has a LU decomposition (after possibly swapping some rows of A first).

- The matrix U is just the echelon form of A produced during Gaussian elimination.
- The matrix L can be constructed, entry-by-entry, by simply recording the row operations used during Gaussian elimination. (No extra work needed!)

Example 8. Determine the LU decomposition of $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution. $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{R_2-3R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ translates into $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (no calculation needed!), we therefore have $A=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Example 9. Determine the LU decomposition of $A=\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix}$.

Solution. We perform Gaussian elimination until we arrive at an echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix} \overset{\substack{R_2-3R_1\Rightarrow R_2 \\ R_3+2R_1\Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 8 & 1 & 3 \end{bmatrix} \overset{R_3+8R_2\Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 9 & -5 \end{bmatrix}$$

Observe that we can reverse both of these steps using the row operations $\begin{array}{c} R_2+3R_1\Rightarrow R_2 \\ R_3-2R_1-8R_2\Rightarrow R_3 \end{array}$.

Encoding these in L, the corresponding LU decomposition of A is

$$A=LU=\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 & 1 \\ & -1 & 1 & -1 \\ & & 9 & -5 \end{bmatrix}.$$

Note that no further computation was required to obtain L. (The entries in the matrix L are precisely the (negative) coefficients in the original row operations.)

Comment. By contrast, combining the operations $\begin{array}{c} R_2 - 3R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3 \end{array}$ and $R_3 + 8R_2 \Rightarrow R_3$ requires computation. That is because we change R_2 in the first step, and then use the changed R_2 in the second step. Indeed, note that

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -3 & 1 & \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ -22 & 8 & 1 \end{bmatrix},$$

so the combined operations are $\begin{array}{c} R_2 - 3R_1 \Rightarrow R_2 \\ R_3 - 22R_1 + 8R_2 \Rightarrow R_3 \end{array}$ (you can also see that directly from the operations).

On the other hand, there was no such complication when combining the reversed operations:

Combining $R_3 - 8R_2 \Rightarrow R_3$ and $\begin{array}{c} R_2 + 3R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3 \end{array}$ simply results in $\begin{array}{c} R_2 + 3R_1 \Rightarrow R_2 \\ R_3 - 2R_1 - 8R_2 \Rightarrow R_3 \end{array}$, as used above.

The difference is that, here, we change R_3 in the first step but then don't use the changed R_3 in the second step. In terms of matrix multiplication, we have

$$\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & -8 & 1 \end{bmatrix},$$

where, because of their special form, the product of the two lower triangular matrices is just "putting together" the entries (unlike in the non-reversed product).

Review. The **RREF** (row-reduced echelon form) of A is obtained from an echelon form by

- scaling the pivots to 1, and then
- eliminating the entries above the pivots.

A typical RREF has the shape [$*$ represents an entry that could be anything]

$$\begin{bmatrix} 1 & * & 0 & * & * & 0 & * \\ & & 1 & * & * & 0 & * \\ & & & & & 1 & * \end{bmatrix}$$

Example 10. Let's compute the RREF of the 3×4 matrix from Example 9.

Solution.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix} \overset{\substack{R_2-3R_1 \Rightarrow R_2 \\ R_3+2R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 8 & 1 & 3 \end{bmatrix} \overset{R_3+8R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 9 & -5 \end{bmatrix}$$

$$\overset{\substack{-R_2 \Rightarrow R_2 \\ \frac{1}{9}R_3 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix} \overset{\substack{R_1-2R_3 \Rightarrow R_1 \\ R_2+R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 0 & \frac{19}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix} \overset{R_1-R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix}$$

Example 11. The RREF of $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ from earlier is the 2×2 identity matrix.

Comment. That's not surprising: A square matrix is invertible if and only if its RREF is the identity matrix. If that isn't obvious to you, think about how you invert a matrix using Gaussian elimination (reviewed next).

Review. Recall the Gauss–Jordan method of computing A^{-1}. Starting with the augmented matrix $[A \mid I]$, we do Gaussian elimination until we obtain the RREF, which will be of the form $[I \mid A^{-1}]$ so that we can read off A^{-1}.

Why does that work? By our discussion, the steps of Gaussian elimination can be expressed by multiplication (on the left) with a matrix B. Only looking at the first part of the augmented matrix, and since the RREF of an invertible matrix is I, we have $BA = I$, which means that we must have $B = A^{-1}$. The other part of the augmented matrix (which is I initially) gets multiplied with $B = A^{-1}$ as well, so that, in the end, it is $BI = A^{-1}$. That's why we can read off A^{-1}!

For instance. To invert $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ using the Gauss–Jordan method, we would proceed as follows:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & -6 & 0 & 1 \end{array}\right] \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -8 & -2 & 1 \end{array}\right] \overset{\substack{\frac{1}{2}R_1 \Rightarrow R_1 \\ -\frac{1}{8}R_2 \Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array}\right] \overset{R_1 - \frac{1}{2}R_2 \Rightarrow R_1}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{8} & \frac{1}{16} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array}\right]$$

We conclude that $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{16} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$.

Of course, for 2×2 matrices it is much simpler to use the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Review: Vector spaces, bases, dimension, null spaces

Review.

- Vectors are things that can be **added** and **scaled**.
- Hence, given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, the most general we can do is form the **linear combination** $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n$. The set of all these linear combinations is the **span** of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, denoted by $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.
- Vector **spaces** are spans.

  Equivalently. Vector spaces are sets of vectors so that the result of adding and scaling remains within that set.

  Homework. Of course, the latter is a very informal statement. Revisit the formal definition, probably consisting of a list of axioms, and observe how that matches with the above (for instance, several of the axioms are concerned with addition and scaling satisfying the "expected" rules).
- Recall that vectors from a vector space V form a **basis** of V if and only if
  - the vectors span V, and
  - the vectors are (linearly) independent.

  Equivalently. $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ from V form a basis of V if and only if every vector in V can be expressed as a unique linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

  Just checking. Make sure that you can define precisely what it means for vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ to be independent.
- The **dimension** of a vector space V is the number of vectors in a basis for V.

  No matter what basis one chooses for V, it always has the same number of vectors.

Example 12. $\mathbb{R}^3$ is the vector space of all vectors with 3 real entries.

$\mathbb{R}$ itself refers to the set of real numbers. We will later also discuss $\mathbb{C}$, the set of complex numbers.

The **standard basis** of $\mathbb{R}^3$ is $\begin{bmatrix} 1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix}$. The dimension of $\mathbb{R}^3$ is 3.

Review. The **null space** $\operatorname{null}(A)$ of a matrix A consists of those vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$.

Make sure that you see why $\operatorname{null}(A)$ is a vector space. [For instance, if you pick two vectors in $\operatorname{null}(A)$ why is it that the sum of them is in $\operatorname{null}(A)$ again?]

Example 13. What is $\operatorname{null}(A)$ if the matrix A is invertible?

Solution. If A is invertible, then $A\boldsymbol{x} = \mathbf{0}$ has the unique solution $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

Hence, $\operatorname{null}(A) = \{\mathbf{0}\}$ which is the trivial vector space (consisting of only the null vector) and has dimension 0.

Example 14. Compute a basis for $\operatorname{null}(A)$ where $A = \begin{bmatrix} -1 & 0 & 2\\ 2 & -3 & 2\\ 1 & 0 & -2 \end{bmatrix}$.

Solution. We perform row operations and obtain

$$\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2\\ 2 & -3 & 2\\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\substack{R_2+2R_1 \Rightarrow R_2\\ R_3+R_1 \Rightarrow R_3}}{=} \operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2\\ 0 & -3 & 6\\ 0 & 0 & 0 \end{bmatrix}\right) \overset{\substack{-R_1 \Rightarrow R_1\\ -\frac{1}{3}R_2 \Rightarrow R_2}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & -2\\ 0 & 1 & -2\\ 0 & 0 & 0 \end{bmatrix}\right).$$

From the RREF, we can now read off the general solution to $A\boldsymbol{x} = \mathbf{0}$:

- x_1 and x_2 are pivot variables. [For each we have an equation expressing it in terms of the other variables; for instance, $x_1 - 2x_3 = 0$ tells us that $x_1 = 2x_3$.]
- x_3 is a free variable. [There is no equation forcing a value on x_3.]
- Hence, without computation, we see that the general solution is $\begin{bmatrix} 2x_3\\ 2x_3\\ x_3 \end{bmatrix}$.

In other words, a basis is $\begin{bmatrix} 2\\2\\1 \end{bmatrix}$.

Comment. We are starting with the three equations $-x_1 + 2x_3 = 0$, $2x_1 - 3x_2 + 2x_3 = 0$, $x_1 - 2x_3 = 0$. Performing row operations on the matrix is the same as combining these equations (with the objective to form simpler equations by eliminating variables).

Example 15. Compute a basis for $\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2\\ 2 & 0 & 2\\ 1 & 0 & 1 \end{bmatrix}\right)$.

Solution.

$$\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2\\ 2 & 0 & 2\\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\substack{R_2-R_1 \Rightarrow R_2\\ R_3-\frac{1}{2}R_1 \Rightarrow R_3}}{=} \operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}\right) \overset{\frac{1}{2}R_1 \Rightarrow R_1}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}\right)$$

This time, x_2 and x_3 are free variables. The general solution is $\begin{bmatrix} -x_3\\ x_2\\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 0\\1\\0 \end{bmatrix} + x_3\begin{bmatrix} -1\\0\\1 \end{bmatrix}$.

Hence, a basis is $\begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\1 \end{bmatrix}$.

Review: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \iff\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \iff\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

More precisely, we find a basis of eigenvectors for the λ-**eigenspace** $\operatorname{null}(A - \lambda I)$.

Example 16. $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ has one eigenvector that is "easy" to see. Do you see it?

Solution. Note that $A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector.

Just for contrast. Note that $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$, where

- the columns of P are the eigenvectors, and
- the diagonal matrix D has the eigenvalues on the diagonal.

Such a diagonalization is possible if and only if A has enough (independent) eigenvectors.

Comment. If you don't quite recall why these choices result in the diagonalization $A = PDP^{-1}$, note that the diagonalization is equivalent to $AP = PD$.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i \implies A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix}$$
$$= \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

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Example 17. Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues and bases for the eigenspaces of A.

(b) Diagonalize A. That is, determine matrices P and D such that $A = PDP^{-1}$.

Solution.

(a) By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda)-2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 5$.

Comment. At this point, we know that we will find one eigenvector for $\lambda = 5$ (more precisely, the 5-eigenspace definitely has dimension 1). On the other hand, the 2-eigenspace might have dimension 2 or 1. In order for A to be diagonalizable, the 2-eigenspace must have dimension 2. (Why?!)

- The 5-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right)$. Proceeding as in Example 14, we obtain

$$\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}.$$

In other words, the 5-eigenspace has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

- The 2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$. Proceeding as in Example 15, we obtain

$$\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

In other words, the 2-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Comment. So, indeed, the 2-eigenspace has dimension 2. In particular, A is diagonalizable.

(b) A possible choice is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Comment. However, many other choices are possible and correct. For instance, the order of the eigenvalues in D doesn't matter (as long as the same order is used for P). Also, for P, the columns can be chosen to be any other set of eigenvectors.

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Example 18. (extra practice) Diagonalize, if possible, the matrices

$$A=\begin{bmatrix}3&4&1\\0&2&0\\1&4&3\end{bmatrix},\quad B=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix},\quad C=\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix}.$$

Solution. For instance, $A=PDP^{-1}$ with $P=\begin{bmatrix}1&-4&-1\\0&1&0\\1&0&1\end{bmatrix}$ and $D=\begin{bmatrix}4&&\\&2&\\&&2\end{bmatrix}$. B is not diagonalizable.

For instance, $C=PDP^{-1}$ with $P=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$ and $D=\begin{bmatrix}1&&\\&0&\\&&0\end{bmatrix}$.

Review: Computing determinants using cofactor expansion

Review. Let A be an $n\times n$ matrix. The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned}\det(A)\neq 0 &\iff A \text{ is invertible}\\ &\iff A\boldsymbol{x}=\boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ (for all } \boldsymbol{b})\\ &\iff A\boldsymbol{x}=0 \text{ is only solved by } \boldsymbol{x}=0\end{aligned}$$

Example 19. $\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$

Example 20. Compute $\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix}=1\cdot\begin{vmatrix}+&&\\&-1&2\\&0&1\end{vmatrix}-2\cdot\begin{vmatrix}&-&\\3&&2\\2&&1\end{vmatrix}+0\cdot\begin{vmatrix}&&+\\3&-1&\\2&0&\end{vmatrix}$$

$$\overset{\text{i.e.}}{=}1\cdot\begin{vmatrix}-1&2\\0&1\end{vmatrix}-2\cdot\begin{vmatrix}3&2\\2&1\end{vmatrix}+0\cdot\begin{vmatrix}3&-1\\2&0\end{vmatrix}=1\cdot(-1)-2\cdot(-1)+0=1$$

Each term in the cofactor expansion is ±1 times an entry times a smaller determinant (row and column of entry deleted).

The ±1 is assigned to each entry according to $\begin{bmatrix}+&-&+&\cdots\\-&+&-&\\+&-&+&\\\vdots&&&\ddots\end{bmatrix}$.

Solution. We expand by the second column:

$$\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix}=-2\cdot\begin{vmatrix}&-&\\3&&2\\2&&1\end{vmatrix}+(-1)\cdot\begin{vmatrix}1&&0\\&+&\\2&&1\end{vmatrix}-0\cdot\begin{vmatrix}1&&0\\3&&2\\&-&\end{vmatrix}$$

$$=-2\cdot(-1)+(-1)\cdot1-0=1$$

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Example 21. Compute $\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix}$.

Solution. We can expand by the second column:

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix} = -0\begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 2 & 8 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

[Of course, you don't have to spell out the 3×3 matrices that get multiplied with 0.]

We can compute the remaining 3×3 matrix in any way we prefer. One option is to expand by the first column:

$$2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} = 2\left(+1\begin{vmatrix} 2 & 1 \\ 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} \right) = 2(1\cdot 2 + 2\cdot(-5)) = -16$$

Comment. For cofactor expansion, choosing to expand by the second column is the best choice because this column has more zeros than any other column or row.

The determinant of a triangular matrix is the product of the diagonal entries.

Why? Can you explain this (you can use the next example) using cofactor expansion?

Example 22. Compute $\begin{vmatrix} 1 & 0 & 3 & -1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$.

Solution. Since the matrix is (upper) triangular, $\begin{vmatrix} 1 & 0 & 3 & -1 \\ & 3 & 1 & 5 \\ & & -2 & 1 \\ & & & 5 \end{vmatrix} = 1\cdot 3\cdot(-2)\cdot 5 = -30$.

Review.

- Effect of row (or column) operations on determinant.
- $\det(AB) = \det(A)\det(B)$
- In particular, the LU decomposition provides us with a way to compute determinants:
  If $A = LU$, then $\det(A) = \det(L)\det(U)$ and the latter determinants are just products of diagonal entries (because both L and U are triangular).

  Comment. Unless a row swap is required, we can compute the LU decomposition of $A = LU$ using only row operations of the form $R_i + cR_j \Rightarrow R_i$ (those don't change the determinant!).
  In that case, the matrix L will have 1's on the diagonal. In particular, $\det(L) = 1$.
  Consequently, in that case, $\det(A) = \det(U)$.

  Practical comment. For larger matrices, cofactor expansion is a terribly inefficient way of computing determinants. Instead, Gaussian elimination (i.e. LU decomposition) is much more efficient.
  On the other hand, cofactor expansion is a good choice when working by hand with small matrices.

Example 23. (review) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Recall that $(AB)^T = B^T A^T$. This reflects the fact that, in the column-centric versus the row-centric interpretation of matrix multiplication, the order of the matrices is reversed.

Comment. When working with complex numbers, the fundamental role is not played by the transpose but by the **conjugate transpose** instead (we'll see that in our discussion of orthogonality): $A^* = \overline{A^T}$.

For instance, if $A = \begin{bmatrix} 1-3i & 5i \\ 2+i & 3 \end{bmatrix}$, then $A^* = \begin{bmatrix} 1+3i & 2-i \\ -5i & 3 \end{bmatrix}$.

Orthogonality

The inner product and distances

Definition 24. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 25. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

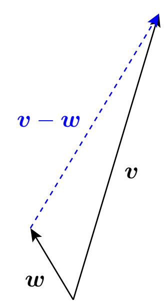
Definition 26.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 27. For instance, in $\mathbb{R}^2$, $\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 28. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

Comment. This is a vector version of $(x-y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

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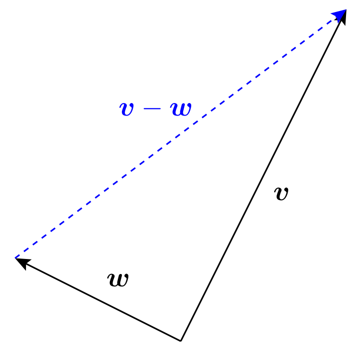
Orthogonal vectors

Definition 29. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$$\iff \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \underbrace{\|\boldsymbol{v} - \boldsymbol{w}\|^2}_{\substack{= \|\boldsymbol{v}\|^2 - 2\boldsymbol{v}\cdot\boldsymbol{w} + \|\boldsymbol{w}\|^2 \\ \text{(by previous example)}}}$$

$$\iff -2\boldsymbol{v} \cdot \boldsymbol{w} = 0$$

$$\iff \boldsymbol{v} \cdot \boldsymbol{w} = 0$$

Definition 30. We say that two subspaces V and W of $\mathbb{R}^n$ are **orthogonal** if and only if every vector in V is orthogonal to every vector in W.

The **orthogonal complement** of V is the space $V^\perp$ of all vectors that are orthogonal to V.

Exercise. Show that the orthogonal complement is indeed a vector space. Alternatively, this follows from our discussion in the next example which leads to Theorem 32. Namely, every space V can be written as $V = \operatorname{col}(A)$ for a suitable matrix A (for instance, we can choose the columns of A to be basis vectors of V). It then follows that $V^\perp = \operatorname{null}(A^T)$ (which is clearly a space).

Example 31. Determine a basis for the orthogonal complement of $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$.

Solution. The orthogonal complement $V^\perp$ consists of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that are orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Using the dot product, this means we must have $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ as well as $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Note that this is equivalent to the equations $1x_1 + 2x_2 + 1x_3 = 0$ and $3x_1 + 1x_2 + 2x_3 = 0$.

In matrix-vector form, these two equations combine to $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This is the same as saying that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ has to be in $\operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \right)$. This means that $V^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \right)$.

[Note that we have done no computations up to this point! Instead, we have derived Theorem 32 below.]

We compute (fill in the work!) that $V^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 1/5 \end{bmatrix} \right) = \operatorname{span}\left\{ \begin{bmatrix} -3/5 \\ -1/5 \\ 1 \end{bmatrix} \right\}$.

Check. $\begin{bmatrix} -3/5 \\ -1/5 \\ 1 \end{bmatrix}$ is indeed orthogonal to both $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Note. If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is orthogonal to both basis vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, then it is orthogonal to every vector in V.

Indeed, vectors in V are of the form $\boldsymbol{v} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and we have $\boldsymbol{v} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{=0} + b \underbrace{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{=0} = 0$.

Just to make sure. Why is it geometrically clear that the orthogonal complement of V is 1-dimensional?

The following theorem follows by the same reasoning that we used in the previous example.

In that example, we started with $V = \mathrm{col}\left(\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ and found that $V^\perp = \mathrm{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right)$.

Theorem 32. If $V = \mathrm{col}(A)$, then $V^\perp = \mathrm{null}(A^T)$.
In particular, if V is a subspace of $\mathbb{R}^n$ with $\dim(V) = r$, then $\dim(V^\perp) = n - r$.

For short. $\mathrm{col}(A)^\perp = \mathrm{null}(A^T)$

Note that the second part can be written as $\dim(V) + \dim(V^\perp) = n$.

To see that this is true, suppose we choose the columns of A to be a basis of V. If V is a subspace of $\mathbb{R}^n$ with $\dim(V) = r$, then A is a $r \times n$ matrix with r pivot columns. Correspondingly, A^T is a $n \times r$ matrix with r pivot rows. Since $n \geqslant r$ there are $n - r$ free variables when computing a basis for $\mathrm{null}(A^T)$. Hence, $\dim(V^\perp) = n - r$.

Example 33. Suppose that V is spanned by 3 linearly independent vectors in $\mathbb{R}^5$. Determine the dimension of V and its orthogonal complement $V^\perp$.

Solution. This means that $\dim V = 3$. By Theorem 32, we have $\dim V^\perp = 5 - 3 = 2$.

Example 34. Determine a basis for the orthogonal complement of (the span of) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Solution. Here, $V = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ and we are looking for the orthogonal complement $V^\perp$.

Since $V = \mathrm{col}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$, it follows from Theorem 32 that $V^\perp = \mathrm{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$.

Computing a basis for $\mathrm{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$ is easy since $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ is already in RREF.

Note that the general solution to $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\boldsymbol{x} = 0$ is $\begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $V^\perp = \mathrm{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$ therefore is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Check. We easily check (do it!) that both of these are indeed orthogonal to the original vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

The fundamental theorem

Review. The four **fundamental subspaces** associated with a matrix A are

$$\mathrm{col}(A),\quad \mathrm{row}(A),\quad \mathrm{null}(A),\quad \mathrm{null}(A^T).$$

Note that $\mathrm{row}(A) = \mathrm{col}(A^T)$. (In particular, we usually write vectors in $\mathrm{row}(A)$ as column vectors.)

Comment. $\mathrm{null}(A^T)$ is called the **left null space** of A.

Why that name? Recall that, by definition $\boldsymbol{x}$ is in $\mathrm{null}(A) \iff A\boldsymbol{x} = \mathbf{0}$.

Likewise, $\boldsymbol{x}$ is in $\mathrm{null}(A^T) \iff A^T\boldsymbol{x} = \mathbf{0} \iff \boldsymbol{x}^T A = \mathbf{0}$.

[Recall that $(AB)^T = B^T A^T$. In particular, $(A^T\boldsymbol{x})^T = \boldsymbol{x}^T A$, which is what we used in the last equivalence.]

Review. The **rank** of a matrix is the number of pivots in its RREF.

Equivalently, as showcased in the next result, the rank is the dimension of either the column or the row space.

Theorem 35. **(Fundamental Theorem of Linear Algebra, Part I)**

Let A be an $m \times n$ matrix of **rank** r.

- $\dim \mathrm{col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \mathrm{row}(A) = r$ (subspace of $\mathbb{R}^n$) $\qquad \mathrm{row}(A) = \mathrm{col}(A^T)$
- $\dim \mathrm{null}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{null}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Example 36. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Solution. Make sure that, for such a simple matrix, you can see all of these that at a glance!

$\mathrm{col}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$, $\mathrm{row}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, $\mathrm{null}(A) = \mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, $\mathrm{null}(A^T) = \mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$

Important observation. The basis vectors for $\mathrm{row}(A)$ and $\mathrm{null}(A)$ are orthogonal! $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Always. Vectors in $\mathrm{null}(A)$ are orthogonal to vectors in $\mathrm{row}(A)$. In short, $\mathrm{null}(A)$ is orthogonal to $\mathrm{row}(A)$.

Why? Suppose that $\boldsymbol{x}$ is in $\mathrm{null}(A)$. That is, $A\boldsymbol{x} = \mathbf{0}$. But think about what $A\boldsymbol{x} = \mathbf{0}$ means (row-product rule). It means that the inner product of every row with $\boldsymbol{x}$ is zero. Which implies that $\boldsymbol{x}$ is orthogonal to the row space.

Theorem 37. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\mathrm{null}(A)$ is orthogonal to $\mathrm{row}(A)$. (both subspaces of $\mathbb{R}^n$)

Note that $\dim \mathrm{null}(A) + \dim \mathrm{row}(A) = n$. Hence, the two spaces are orthogonal complements.

- $\mathrm{null}(A^T)$ is orthogonal to $\mathrm{col}(A)$.

Again, the two spaces are orthogonal complements. (This is just the first part with A replaced by A^T.)

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Example 38. Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$. Check that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & -3 & -9 \end{bmatrix} \underset{\rightsquigarrow}{R_3 - \frac{3}{2}R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\rightsquigarrow}{-\frac{1}{2}R_2 \Rightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{R_1 - R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$, $\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

$\operatorname{null}(A)$ and $\operatorname{row}(A)$ are indeed orthogonal, as certified by:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$$

In fact, $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements because the dimensions add up to $2 + 2 = 4$.

In particular, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ form a basis of all of $\mathbb{R}^4$.

Example 39. (extra) Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ (as well as $\operatorname{null}(A^T)$ and $\operatorname{col}(A)$) are orthogonal complements.

Partial solution. One can almost see that $\operatorname{rank}(A) = 3$. Hence, the dimensions of the fundamental subspaces are ...

Consistency of a system of equations

Example 40. (warmup) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Note that this means that the system of equations $\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 1 \\ 3x_1 & + & x_2 & = & 1 \\ & & 5x_2 & = & 1 \end{array}$ can also be written as $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

[This was the motivation for introducing matrix-vector multiplication.]

In the same way, any system can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where A is a matrix and $\boldsymbol{b}$ a vector.
In particular, this makes it obvious that:

$$A\boldsymbol{x} = \boldsymbol{b} \text{ is consistent} \quad \Longleftrightarrow \quad \boldsymbol{b} \text{ is in } \operatorname{col}(A)$$

Recall that, by the FTLA, $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$ are orthogonal complements.

Theorem 41. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\quad \Longleftrightarrow \quad$ $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$

Proof. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\Longleftrightarrow$ $\boldsymbol{b}$ is in $\operatorname{col}(A)$ $\overset{\text{FTLA}}{\Longleftrightarrow}$ $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$

Note. $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$ means that $\boldsymbol{y}^T\boldsymbol{b} = 0$ whenever $\boldsymbol{y}^T A = \boldsymbol{0}$. Why?!

Example 42. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

Solution. (old)

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 1 & b_2 \\ 0 & 5 & b_3 \end{array}\right] \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 5 & b_3 \end{array}\right] \overset{R_3 + R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

So, $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $-3b_1 + b_2 + b_3 = 0$.

Solution. (new) We determine a basis for $\operatorname{null}(A^T)$:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -5 & 5 \end{bmatrix} \overset{-\frac{1}{5}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix} \overset{R_1 - 3R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

We read off from the RREF that $\operatorname{null}(A^T)$ has basis $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

$\boldsymbol{b}$ has to be orthogonal to $\operatorname{null}(A^T)$. That is, $\boldsymbol{b} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$. As above!

Comment. Below is how we can use Sage to (try and) solve $A\boldsymbol{x}=\boldsymbol{b}$ for $\boldsymbol{b}=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{b}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

```
>>> A = matrix([[1,2],[3,1],[0,5]])
>>> A.solve_right(vector([1,1,2]))
```

$$\left(\frac{1}{5}, \frac{2}{5}\right)$$

```
>>> A.solve_right(vector([1,1,1]))

ValueError: matrix equation has no solutions

During handling of the above exception, another exception occurred:

ValueError: matrix equation has no solutions
```

Least squares

Example 43. Not all linear systems have solutions.

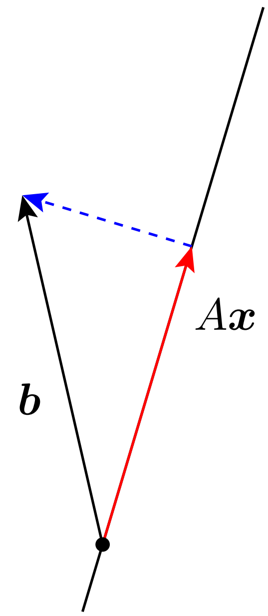
In fact, for many applications, data needs to be fitted and there is no hope for a perfect match.

For instance, $A\boldsymbol{x} = \boldsymbol{b}$ with

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has no solution:

- $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in $\operatorname{col}(A)$ since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \neq 0$ (see previous example).
- Instead of giving up, we want the $\boldsymbol{x}$ which makes $A\boldsymbol{x}$ and $\boldsymbol{b}$ as close as possible.
- Such $\boldsymbol{x}$ is characterized by the error $A\boldsymbol{x} - \boldsymbol{b}$ being **orthogonal** to $\operatorname{col}(A)$ (i.e. all possible $A\boldsymbol{x}$).



Definition 44. $\hat{\boldsymbol{x}}$ is a **least squares solution** of the system $A\boldsymbol{x} = \boldsymbol{b}$ if $\hat{\boldsymbol{x}}$ is such that $A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is as small as possible (i.e. minimal norm).

- If $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, then $\hat{\boldsymbol{x}}$ is just an ordinary solution. (in that case, $A\hat{\boldsymbol{x}} - \boldsymbol{b} = \boldsymbol{0}$)
- Interesting case: $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent. (in particular, if the system is overdetermined)

The normal equations

The following result provides a straightforward recipe (thanks to the FTLA) to find least squares solutions for all systems $A\boldsymbol{x} = \boldsymbol{b}$.

Theorem 45. $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$
$\iff A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (the **normal equations**)

Proof.

$\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$

$\iff A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is as small as possible

$\iff A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is orthogonal to $\operatorname{col}(A)$

$\overset{\text{FTLA}}{\iff} A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is in $\operatorname{null}(A^T)$

$\iff A^T(A\hat{\boldsymbol{x}} - \boldsymbol{b}) = \boldsymbol{0}$

$\iff A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ □

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Example 46. Find the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. First, $A^T A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Hence, the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ take the form $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
Solving, we immediately find $\hat{\boldsymbol{x}} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$.

Check. Since $A\hat{\boldsymbol{x}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, the error is $A\hat{\boldsymbol{x}} - \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. Recall that the error must be orthogonal to $\operatorname{col}(A)$!

This error is indeed orthogonal to $\operatorname{col}(A)$ because $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$.

Comment. Why are the normal equations so particularly simple (compare with example below for the typical case) here? Note how each entry of the product $A^T A$ is computed as the dot product of two columns of A (matrix products of a row of A^T times a column of A). That $A^T A$ is a diagonal matrix reflects the fact that the two columns of A are orthogonal to each other.

Example 47. Find the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. First, $A^T A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 30 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

Hence, the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ take the form $\begin{bmatrix} 10 & 5 \\ 5 & 30 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

Since $\begin{bmatrix} 10 & 5 \\ 5 & 30 \end{bmatrix}^{-1} = \frac{1}{275} \begin{bmatrix} 30 & -5 \\ -5 & 10 \end{bmatrix} = \frac{1}{55} \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix}$, we find $\hat{\boldsymbol{x}} = \frac{1}{55} \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{55} \begin{bmatrix} 16 \\ 12 \end{bmatrix}$.

Check. Since $A\hat{\boldsymbol{x}} = \frac{1}{55} \begin{bmatrix} 40 \\ 60 \\ 60 \end{bmatrix}$, the error $A\hat{\boldsymbol{x}} - \boldsymbol{b} = \frac{1}{55} \begin{bmatrix} -15 \\ 5 \\ 5 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ must be orthogonal to $\operatorname{col}(A)$.

The error is indeed orthogonal to $\operatorname{col}(A)$ because $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \cdot \frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \cdot \frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$.

Any serious linear algebra problems are done by a machine. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at sagemath.org. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at cocalc.com from any browser. For short computations, like the one below, you can also just use the input field on our course website.

Sage is built as a **Python** library, so any Python code is valid. Here, we will just use it as a fancy calculator.

Let's revisit Example 38 and let Sage do the work for us:

```
>>> A = matrix([[1,2,1,4],[2,4,0,2],[3,6,0,3]])
```

```
>>> A.rref()
```

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly, if we wanted to compute a basis for $\mathrm{null}(A^T)$, we can simply do:

```
>>> A.transpose().rref()
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here are some other standard things we might be interested in (compare with Example 17):

```
>>> A = matrix([[4,0,2],[2,2,2],[1,0,3]])
```

```
>>> A.eigenvalues()
```

$$[5, 2, 2]$$

```
>>> A.eigenvectors_right()
```

$$\left[\left(5, \left[\left(1,\ 1,\ \frac{1}{2}\right)\right], 1\right), (2, [(1,\ 0,\ -1), (0,\ 1,\ 0)], 2)\right]$$

```
>>> A.eigenmatrix_right()
```

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1}{2} & -1 & 0 \end{bmatrix}\right)$$

```
>>> A.rank()
```

3

```
>>> A.determinant()
```

$$20$$

```
>>> A.inverse()
```

$$\begin{pmatrix} \frac{3}{10} & 0 & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{2} & -\frac{1}{5} \\ -\frac{1}{10} & 0 & \frac{2}{5} \end{pmatrix}$$

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Application: least squares lines

Given data points (x_i, y_i), we wish to find optimal parameters a, b such that $y_i \approx a + bx_i$ for all i.

Example 48. Determine the line that "best fits" the data points $(2,1), (5,2), (7,3), (8,3)$.

Comment. Can you see that there is no line fitting the data perfectly? (Check out the last two points!)

Solution. We need to determine the values a, b for the best-fitting line $y = a + bx$.
If there was a line that fit the data perfectly, then:

$$\begin{aligned} a + 2b &= 1 \qquad (2,1) \\ a + 5b &= 2 \qquad (5,2) \\ a + 7b &= 3 \qquad (7,3) \\ a + 8b &= 3 \qquad (8,3) \end{aligned}$$

In matrix form, this is: $\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix}}_{\text{design matrix} X} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\text{observation vector } \boldsymbol{y}}$ (writing the points as (x_i, y_i))

Using our points, these equations become $\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$. [This system is inconsistent (as expected).]

We compute a least squares solution.

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}, \qquad X^T \boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}.$$

Solving the normal equations $\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$, we find $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}$.

Hence, the least squares line is $y = \frac{2}{7} + \frac{5}{14} x$.

The plot above shows our points together with this line. It does look like a very good fit!

Important comment. In what sense is this the line of "best fit"? By computing a least squares solution the way we do, we are minimizing the error $\boldsymbol{y} - X \begin{bmatrix} a \\ b \end{bmatrix}$. The components of that error are $y_i - (a + bx_i)$.

Hence, we see that we are minimizing the **residual sum of squares** $\mathrm{SS}_{\mathrm{res}} = \sum_i [y_i - (a + bx_i)]^2$.

Also see the discussion after the next example (where we swap the role of x and y) as well as the example at the beginning of next class (where we discuss making predictions and why minimizing $\mathrm{SS}_{\mathrm{res}}$ corresponds to minimizing the error of those predictions).

Example 49. (again) Determine the least squares line for the points $(2,1),(5,2),(7,3),(8,3)$.

Solution. Let's repeat the computation we did in the previous example. This time, we let Sage do the actual work for us:

```
>>> X = matrix([[1,2],[1,5],[1,7],[1,8]]); y = vector([1,2,3,3])
>>> (X.transpose()*X).solve_right(X.transpose()*y)
```

$$\left(\frac{2}{7}, \frac{5}{14}\right)$$

Here are some intermediate steps to help see what's going on (and that it matches our earlier work):

```
>>> X.transpose()*X
```

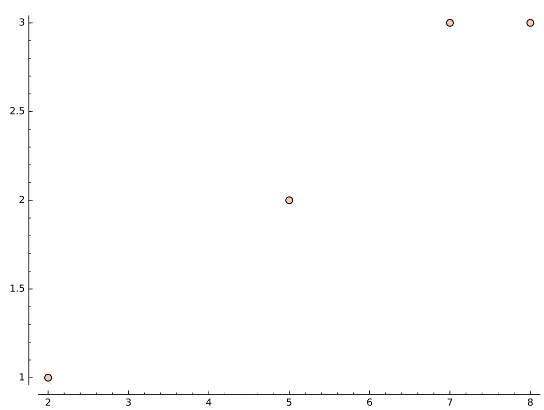
$$\begin{pmatrix} 4 & 22 \\ 22 & 142 \end{pmatrix}$$

```
>>> X.transpose()*y
```

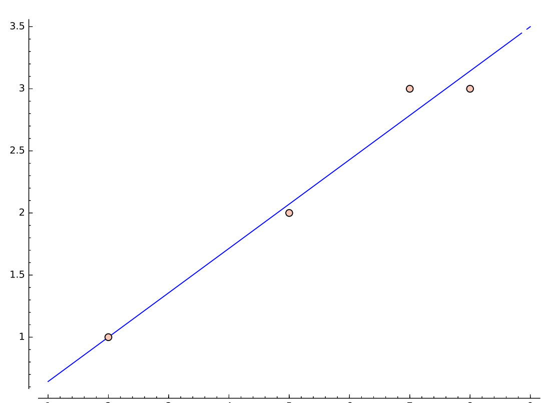
$$(9, 57)$$

Let's plot the least squares line $y = \frac{2}{7} + \frac{5}{14}x$ in Sage to marvel at the good fit!

```
>>> points = [[2,1],[5,2],[7,3],[8,3]]
>>> scatter_plot(points)
```




```
>>> scatter_plot(points) + plot(2/7+5/14*x,1,9)
```



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Comment. As mentioned earlier, the least squares line minimizes the (sum of squares of the) vertical offsets:

http://mathworld.wolfram.com/LeastSquaresFitting.html

Comment. We get a (slightly) different "best fit" line if we change the role of x and y! Can you explain that?

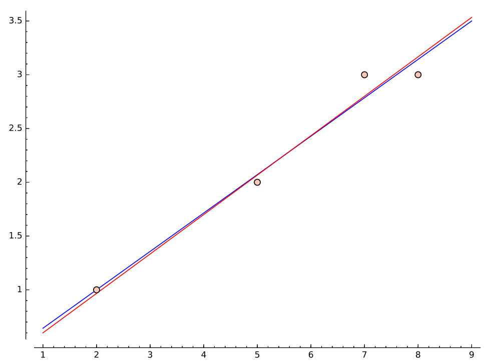
```
>>> X = matrix([[1,1],[1,2],[1,3],[1,3]]); y = vector([2,5,7,8])
```

```
>>> (X.transpose()*X).solve_right(X.transpose()*y)
```

$$\left(-\frac{7}{11}, \frac{30}{11}\right)$$

Note that $x = -\frac{7}{11} + \frac{30}{11}y$ is equivalent to $y = \frac{7}{30} + \frac{11}{30}x$.

```
>>> scatter_plot([[2,1],[5,2],[7,3],[8,3]]) + plot(2/7+5/14*x,1,9) + plot(7/30+11/30*x,1,
    9,color='red')
```



The explanation is that (see pictures at the beginning of this example) we are minimizing vertical offsets in one case and horizontal offsets in the other case.

In linear regression, the relationship between a dependent variable and one or more explanatory variables is modeled. If y is the dependent variable, with x the explanatory variable, then it is natural to minimize the error we make in "predicting y through x" (vertical offsets). See next example.

Example 50. A car rental company wants to predict the annual maintenance cost y (in 100USD/year) of a car using the age x (in years) of that car (as an explanatory variable). Based on the observations $(x, y) = (2, 1), (5, 2), (7, 3), (8, 3)$, predict the cost for a 4.5 year old car (using linear regression).

Solution. Once we compute the regression line $y = a + bx$ (we already did that: $y = \frac{2}{7} + \frac{5}{14}x$), our prediction is $\frac{2}{7} + \frac{5}{14} \cdot 4.5 = \frac{53}{28} \approx 1.89$, that is, 189 USD/year.

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Application: multiple linear regression

> *In statistics, **linear regression** is an approach for modeling the relationship between a scalar dependent variable and one or more explanatory variables.*
>
> *The case of one explanatory variable is called simple linear regression.*
>
> *For more than one explanatory variable, the process is called multiple linear regression.*
>
> http://en.wikipedia.org/wiki/Linear_regression

The experimental data might be of the form (x_i, y_i, z_i), where now the dependent variable z_i depends on two explanatory variables x_i, y_i (instead of just x_i).

Example 51. Set up a linear system to find values for the parameters a, b, c such that $z = a + bx + cy$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$

Solution. The equations $a + bx_i + cy_i = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\text{design matrix } A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\substack{\text{observation} \\ \text{vector} \boldsymbol{z}}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution by solving $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \boldsymbol{z}$.

Application: Fitting data to other curves

We can also fit the experimental data (x_i, y_i) using other curves.

Example 52. Set up a linear system to find values for the parameters a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits some given points $(x_1, y_1), (x_2, y_2), \ldots$

Solution. $y_i \approx a + bx_i + cx_i^2$ with parameters a, b, c.

The equations $y_i = a + bx_i + cx_i^2$ in matrix form:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\text{design matrix} A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\substack{\text{observation} \\ \text{vector } \boldsymbol{y}}}$$

Again, we determine values for a, b, c by computing a least squares solution to that system.

That is, we need to solve the system $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \boldsymbol{y}$.

Example 53. (homework) Use Sage to find values for a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits the points $(0,1), (1,2), (2,3), (3,-4), (4,-7), (5,-12)$.

Solution. We first input the points:

```
>>> points = [[0,1],[1,2],[2,3],[3,-4],[4,-7],[5,-12]]
```

We set up the system described in the previous example, then determine a least-squares solution.

```
>>> X = matrix([[1,0,0],[1,1,1],[1,2,4],[1,3,9],[1,4,16],[1,5,25]])
```

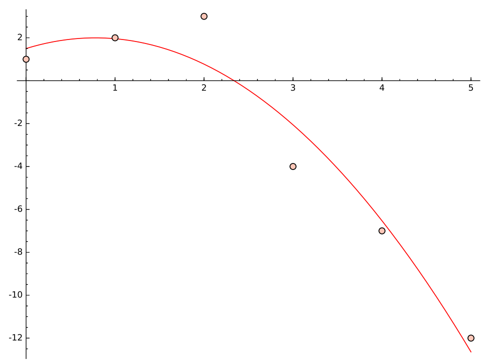
```
>>> y = vector([1,2,3,-4,-7,-12])
```

```
>>> (X.transpose()*X).solve_right(X.transpose()*y)
```

$$\left(\frac{3}{2}, \frac{179}{140}, -\frac{23}{28}\right)$$

Hence, the best fitting quadratic curve is $y = \frac{3}{2} + \frac{179}{140} - \frac{23}{28}x^2$. Here's a plot:

```
>>> scatter_plot(points) + plot(3/2+179/140*x-23/28*x^2,0,5,color='red')
```



Advanced comment. If you are comfortable with Python, you can avoid typing out X and $\boldsymbol{y}$:
[The plot command above now won't work anymore because we are overwriting x with numbers.]

```
>>> X = matrix([[1,x,x^2] for x,y in points])
```

```
>>> y = vector([y for x,y in points])
```

More on orthogonality

Example 54. (review) Find the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Solution. First, $A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$ and $A^T\boldsymbol{b} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$.

Hence, the normal equations $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ take the form $\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}\hat{\boldsymbol{x}} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$. Solving, we find $\hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Check. The error $A\hat{\boldsymbol{x}} - \boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix}$ is indeed orthogonal to $\operatorname{col}(A)$. Because $\begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$.

Orthogonal projections

The **(orthogonal) projection** $\hat{\boldsymbol{b}}$ of a vector $\boldsymbol{b}$ onto a subspace W is the vector in W closest to $\boldsymbol{b}$.

We can compute $\hat{\boldsymbol{b}}$ as follows:

- Write $W = \operatorname{col}(A)$ for some matrix A.
- Then $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ where $\hat{\boldsymbol{x}}$ is a least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$. (i.e. $\hat{\boldsymbol{x}}$ solves $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$)

Why? Why is $A\hat{\boldsymbol{x}}$ the projection of $\boldsymbol{b}$ onto $\operatorname{col}(A)$?

Because, if $\hat{\boldsymbol{x}}$ is a least squares solution then $A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is as small as possible (and any element in $\operatorname{col}(A)$ is of the form $A\boldsymbol{x}$ for some $\boldsymbol{x}$).

Note. This is a recipe for computing any orthogonal projection! That's because every subspace W can be written as $\operatorname{col}(A)$ for some choice of the matrix A (take, for instance, A so that its columns are a basis for W).

Assuming A^TA is invertible (which, as discussed in the lemma below, is automatically the case if the columns of A are independent), we have $\hat{\boldsymbol{x}} = (A^TA)^{-1}A^T\boldsymbol{b}$ and hence:

(projection matrix) The projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto $\operatorname{col}(A)$ is (assuming cols of A are independent)

$$\hat{\boldsymbol{b}} = \underbrace{A(A^TA)^{-1}A^T}_{P}\boldsymbol{b}.$$

The matrix $P = A(A^TA)^{-1}A^T$ is the **projection matrix** for projecting onto $\operatorname{col}(A)$.

Example 55.

(a) What is the orthogonal projection of $\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$?

(b) What is the matrix P for projecting onto $W = \operatorname{span}\left\{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$?

(c) **(once more)** Using P, what is the orthogonal projection of $\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ onto W?

(d) Using P, what is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ onto W?

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Solution.

(a) In other words, what is the orthogonal projection of $\boldsymbol{b}=\begin{bmatrix}2\\0\\11\end{bmatrix}$ onto $\operatorname{col}(A)$ with $A=\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}$.

In Example 54, we found that the system $A\boldsymbol{x}=\boldsymbol{b}$ has the least squares solution $\boldsymbol{x}=\begin{bmatrix}1\\2\end{bmatrix}$.

The projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto $\operatorname{col}(A)$ thus is $A\hat{\boldsymbol{x}}=\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}4\\0\\1\end{bmatrix}+2\begin{bmatrix}0\\2\\1\end{bmatrix}=\begin{bmatrix}4\\4\\3\end{bmatrix}$.

Check. The error $\hat{\boldsymbol{b}}-\boldsymbol{b}=\begin{bmatrix}2\\4\\-8\end{bmatrix}$ needs to be orthogonal to $\operatorname{col}(A)$. Indeed: $\begin{bmatrix}2\\4\\-8\end{bmatrix}\cdot\begin{bmatrix}4\\0\\1\end{bmatrix}=0$ and $\begin{bmatrix}2\\4\\-8\end{bmatrix}\cdot\begin{bmatrix}0\\2\\1\end{bmatrix}=0$.

(b) Note that $W=\operatorname{col}(A)$ for $A=\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}$ and that $A^TA=\begin{bmatrix}17&1\\1&5\end{bmatrix}$. Thus $(A^TA)^{-1}=\frac{1}{84}\begin{bmatrix}5&-1\\-1&17\end{bmatrix}$.

$$P=A(A^TA)^{-1}A^T=\frac{1}{84}\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}\begin{bmatrix}5&-1\\-1&17\end{bmatrix}\begin{bmatrix}4&0&1\\0&2&1\end{bmatrix}=\frac{1}{21}\begin{bmatrix}20&-2&4\\-2&17&8\\4&8&5\end{bmatrix}$$

(c) The orthogonal projection of $\begin{bmatrix}2\\0\\11\end{bmatrix}$ onto W is $P\begin{bmatrix}2\\0\\11\end{bmatrix}=\frac{1}{21}\begin{bmatrix}20&-2&4\\-2&17&8\\4&8&5\end{bmatrix}\begin{bmatrix}2\\0\\11\end{bmatrix}=\frac{1}{21}\begin{bmatrix}84\\84\\63\end{bmatrix}=\begin{bmatrix}4\\4\\3\end{bmatrix}$.

Note. Of course, that agrees with what our computations in the first part. Note that computing P is more work than what we did in in the first part. However, after having computed P once, we can easily project many vectors onto W.

(d) The orthogonal projection of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ onto W is $P\begin{bmatrix}1\\0\\0\end{bmatrix}=\frac{1}{21}\begin{bmatrix}20&-2&4\\-2&17&8\\4&8&5\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\frac{1}{21}\begin{bmatrix}20\\-2\\4\end{bmatrix}$.

Check. The error $\begin{bmatrix}1\\0\\0\end{bmatrix}-\frac{1}{21}\begin{bmatrix}20\\-2\\4\end{bmatrix}=\frac{1}{21}\begin{bmatrix}1\\2\\-4\end{bmatrix}$ is indeed orthogonal to both $\begin{bmatrix}4\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\2\\1\end{bmatrix}$.

Example 56. (extra)

(a) What is the orthogonal projection of $\begin{bmatrix}1\\2\\2\end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$?

(b) What is the orthogonal projection of $\begin{bmatrix}1\\2\\2\end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$?

Solution. (final answer only) The projections are $\left(\frac{11}{6},\frac{1}{3},\frac{7}{6}\right)^T$ and $\left(\frac{3}{2},0,\frac{3}{2}\right)^T$.

Example 57. (extra)

(a) What is the matrix P for projecting onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$?

(b) Using the projection matrix, project $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Solution.

(a) Choosing $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, the projection matrix P is $A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 2 & -4 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Comment. We can choose A in any way such that its columns are a basis for W. The final projection matrix will always be the same.

(b) The projection is $\frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$.

Check. The error $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$ is indeed orthogonal to W.

Lemma 58. If the columns of a matrix A are independent, then A^TA is invertible.

Proof. Assume A^TA is not invertible, so that $A^TA\boldsymbol{x} = \mathbf{0}$ for some $\boldsymbol{x} \neq \mathbf{0}$. Multiply both sides with $\boldsymbol{x}^T$ to get

$$\boldsymbol{x}^TA^TA\boldsymbol{x} = (A\boldsymbol{x})^TA\boldsymbol{x} = \|A\boldsymbol{x}\|^2 = 0,$$

which implies that $A\boldsymbol{x} = 0$. Since the columns of A are independent, this shows that $\boldsymbol{x} = \mathbf{0}$. A contradiction! □

Example 59. If P is a projection matrix, then what is P^2?

For instance. For P as in Example 57, $P^2 = \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}^2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = P$.

Solution. Can you see why it is always true that $P^2 = P$?

[Recall that P projects a vector onto a space W (actually, $W = \operatorname{col}(P)$). Hence P^2 takes a vector $\boldsymbol{b}$, projects it onto W to get $\hat{\boldsymbol{b}}$, and then projects $\hat{\boldsymbol{b}}$ onto W again. But the projection of $\hat{\boldsymbol{b}}$ onto W is just $\hat{\boldsymbol{b}}$ (why?!), so that P^2 always has the exact same effect as P. Therefore, $P^2 = P$.]

Example 60. True or false? If P is the matrix for projecting onto W, then $W = \operatorname{col}(P)$.

Solution. True!

Why? The columns of P are the projections of the standard basis vectors and hence in W. On the other hand, for any vector $\boldsymbol{w}$ in W, we have $P\boldsymbol{w} = \boldsymbol{w}$ so that $\boldsymbol{w}$ is a combination of the columns of P.

[This may take several readings to digest but do read (or ask) until it makes sense!]

In particular. $\operatorname{rank}(P) = \dim W$ (because, for any matrix, $\operatorname{rank}(A) = \dim \operatorname{col}(A)$)

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Review. The **projection matrix** for projecting onto $\mathrm{col}(A)$ is $P = A(A^T A)^{-1} A^T$.

Projecting onto 1-dimensional spaces

When we project onto a 1-dimensional space $\mathrm{span}\{\boldsymbol{w}\}$, we usually just say that we are projecting onto $\boldsymbol{w}$.

The (orthogonal) projection of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is $\dfrac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2}\boldsymbol{w}$.

Why? Replace $\boldsymbol{b}$ with $\boldsymbol{v}$ and A with $\boldsymbol{w}$ in our general projection matrix formula to get $\boldsymbol{w}(\boldsymbol{w}^T\boldsymbol{w})^{-1}\boldsymbol{w}^T\boldsymbol{v}$, which equals $\dfrac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2}\boldsymbol{w}$ (note that $\boldsymbol{w}^T\boldsymbol{v} = \boldsymbol{w}\cdot\boldsymbol{v}$ and $\boldsymbol{w}^T\boldsymbol{w} = \|\boldsymbol{w}\|^2$ are scalars).

Comment. If you have taken Calculus 3, you have seen that formula before. Most likely, you were deriving it using angles at that time. Namely, the dot product has the following connection to angles:

$\boldsymbol{v}\cdot\boldsymbol{w} = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta$ where $\theta\in[0,\pi]$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Why? You can derive this by repeating what we did, right after Definition 29 to show that $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal if and only if $\boldsymbol{v}\cdot\boldsymbol{w}=0$. Just replace Pythagoras with the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$ holds in any triangle!).

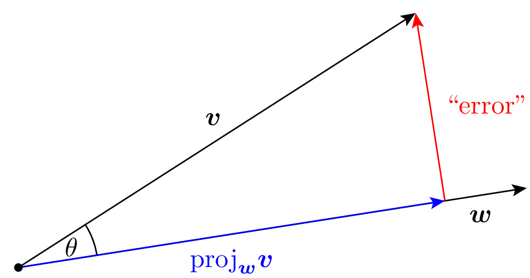
Two obvious cases. Observe that the cases $\theta = 0$ and $\theta = 90°$ are clearly true.

We will not discuss angles much further in this class. Just in case it is helpful, here is the typical argument given in Calculus 3 to determine the projection $\mathrm{proj}_{\boldsymbol{w}}\boldsymbol{v}$ of $\boldsymbol{v}$ onto $\boldsymbol{w}$:

From the sketch, we see that "error" $= \boldsymbol{v} - \mathrm{proj}_{\boldsymbol{w}}\boldsymbol{v}$ and that this error is orthogonal to $\boldsymbol{w}$.

Basic trigonometry tells us that the length of $\mathrm{proj}_{\boldsymbol{w}}\boldsymbol{v}$ is $\|\boldsymbol{v}\|\cos\theta$. Hence:

$$\mathrm{proj}_{\boldsymbol{w}}\boldsymbol{v} = \underbrace{\|\boldsymbol{v}\|\cos\theta}_{\text{length}}\ \underbrace{\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}}_{\text{direction}} = \frac{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta}{\|\boldsymbol{w}\|}\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{w}\|^2}\right)\boldsymbol{w}$$



Orthogonal bases

Review. Vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ are a **basis** for V.

$\Longleftrightarrow$ $V = \mathrm{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n\}$ and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ are linearly independent.

$\Longleftrightarrow$ Any vector $\boldsymbol{w}$ in V can be written as $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n$ in a unique way.

The latter is the practical reason why we care so much about bases!

V could be some abstract vector space (of polynomials or Fourier series), meaning that vectors are abstract objects and not just our usual column vectors. However, as soon as we pick a basis of V, then we can represent every (abstract) vector $\boldsymbol{w}$ by the (usual) column vector $(c_1, c_2, \ldots, c_n)^T$.

This means all of our results can be used, too, when working with these abstract spaces!

Definition 61. A basis $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ of a vector space V is an **orthogonal basis** if the vectors are (pairwise) orthogonal. If, in addition, the basis vectors have length 1, then this is called an **orthonormal basis**.

Example 62. The standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^3$.

Example 63. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$? Is it orthonormal?

Solution. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$.

So, this is an orthogonal basis.

On the other hand, the vectors do not all have length 1, so that this basis is not orthonormal.

Note. Orthogonal vectors are always linearly independent (see next class). Here, this certifies that the three vectors are linearly independent (and hence a basis for $\mathbb{R}^3$).

Normalize the vectors to produce an orthonormal basis.

Solution.

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} = \sqrt{2} \implies$ normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \sqrt{2} \implies$ normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = 1 \implies$ is already normalized: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The resulting orthonormal basis is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Theorem 64. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal. Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.

Proof. Suppose that $c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}$. In order to show that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are independent, we need to show that $c_1 = c_2 = \ldots = c_n = 0$.

Take the dot product of $\boldsymbol{v}_1$ with both sides:

$$\begin{aligned} 0 &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = c_1\|\boldsymbol{v}_1\|^2 \end{aligned}$$

But $\|\boldsymbol{v}_1\| \neq 0$ and hence $c_1 = 0$. Likewise, we find $c_2 = 0$, …, $c_n = 0$. Hence, the vectors are independent. □

Comment. Note that this result is intuitively obvious: if the vectors were linearly dependent, then one of them could be written as a linear combination of the others. However, all these other vectors (and hence any combination of them) are orthogonal to it.

Orthogonal projections if we have an orthogonal basis

Lemma 65. (orthogonal projection if we have an orthogonal basis)
If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthogonal, then the orthogonal projection of $\boldsymbol{w}$ onto $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is

$$\hat{\boldsymbol{w}} = \underbrace{\frac{\boldsymbol{w} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_1}} \boldsymbol{v}_1 + \ldots + \underbrace{\frac{\boldsymbol{w} \cdot \boldsymbol{v}_n}{\boldsymbol{v}_n \cdot \boldsymbol{v}_n}}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_n}} \boldsymbol{v}_n.$$

Proof. It suffices to show that the error $\boldsymbol{w} - \hat{\boldsymbol{w}}$ is orthogonal to each $\boldsymbol{v}_i$. Indeed:

$$(\boldsymbol{w} - \hat{\boldsymbol{w}}) \cdot \boldsymbol{v}_i = \left(\boldsymbol{w} - \frac{\boldsymbol{w} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 - \ldots - \frac{\boldsymbol{w} \cdot \boldsymbol{v}_n}{\boldsymbol{v}_n \cdot \boldsymbol{v}_n} \boldsymbol{v}_n\right) \cdot \boldsymbol{v}_i = \boldsymbol{w} \cdot \boldsymbol{v}_i - \frac{\boldsymbol{w} \cdot \boldsymbol{v}_i}{\boldsymbol{v}_i \cdot \boldsymbol{v}_i} \boldsymbol{v}_i \cdot \boldsymbol{v}_i = 0.$$

Alternatively, can you deduce the formula (say, in the case of an orthonormal basis) from our earlier formula for the projection matrix? □

Important consequence. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}.$$

If the $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a basis, but not orthogonal, then we have to solve a system of equations to find the c_i. That is a lot more work than simply computing a few dot products.

Note. In other words, $\boldsymbol{w}$ decomposes as the sum of its projections onto each basis vector.

Note. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthonormal, then the denominators are all 1.

Example 66. What is the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$?

Comment. We know how to do this using least squares. (Do it for practice!)
However, realizing that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal makes things easier.
[Actually, here, it is obvious what the projection is going to be if we realized that W is the x-y-plane.]

Solution. (using orthogonality) Because $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal, the projection is

$$\underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{projection onto } \boldsymbol{v}_1} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{projection onto } \boldsymbol{v}_2} = \frac{-4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}.$$

Important note. Note that, at this point, we can easily extend $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to an orthogonal basis of $\mathbb{R}^3$:

That is because the error $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ is orthogonal to both of the existing basis vectors.

Therefore $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ is an orthogonal basis of $\mathbb{R}^3$.

This observation underlies the Gram-Schmidt process, which we will discuss next class.

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Example 67. Express $\underbrace{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}}_{\boldsymbol{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{v}_3}$.

Solution. Because $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is an orthogonal basis of $\mathbb{R}^3$, we get (much as in the previous example):

$$\begin{aligned}
\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_1} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_2} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_3} \\
&= \frac{-4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

Because we spelled out all the details this looks more involved than it is. We only computed 6 dot products!

Alternative. We could have solved $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ to also find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$.

The numbers are particularly easy here but in general, to find this solution, we have to go through the entire process of Gaussian elimination. On the other hand, if we have an orthogonal basis, the former approach requires less work, because it is just computing a few dot products.

Example 68. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Solution. This is not an orthogonal basis, so we cannot proceed as in the previous example.

To write $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, we need to solve $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$.

Solving that system (do it!), we find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$.

Review. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthogonal, the orthogonal projection of $\boldsymbol{w}$ onto $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is

$$\hat{\boldsymbol{w}} = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 + \ldots + \frac{\boldsymbol{w} \cdot \boldsymbol{v}_n}{\boldsymbol{v}_n \cdot \boldsymbol{v}_n} \boldsymbol{v}_n.$$

Example 69.

(a) Project $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$.

(b) Express $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

Solution.

(a) We note that the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ are orthogonal to each other.

Therefore, the projection can be computed as $\dfrac{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \dfrac{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{8}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Comment. If we didn't have an orthogonal basis for $W = \operatorname{col}\left(\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \right)$, then we would have to solve the least squares problem $\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ instead to get the same final result (with more work).

(b) Note that this basis is orthogonal! Therefore, we can compute $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{8}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \frac{5}{30} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

(We proceed exactly as in the previous part to compute each coefficient as a quotient of dot products.)

Gram–Schmidt

(Gram–Schmidt orthogonalization)

Given a basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ for W, we produce an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ for W as follows:

- $\boldsymbol{q}_1 = \boldsymbol{w}_1$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_2 \text{ onto } \boldsymbol{q}_1 \end{array} \right)$
- $\boldsymbol{q}_3 = \boldsymbol{w}_3 - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_1 \end{array} \right) - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_2 \end{array} \right)$
- $\boldsymbol{q}_4 = \ldots$

Note. Since $\boldsymbol{q}_1, \boldsymbol{q}_2$ are orthogonal, $\left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\} \end{array} \right) = \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_1 \end{array} \right) + \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_2 \end{array} \right)$.

Important comment. When working numerically on a computer it actually saves time to compute an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ by the same approach but always normalizing each $\boldsymbol{q}_i$ along the way. The reason this saves time is that now the projections onto $\boldsymbol{q}_i$ only require a single dot product (instead of two). This is called **Gram–Schmidt orthonormalization**. When working by hand, it is usually simpler to wait until the end to normalize (so as to avoid working with square roots).

Note. When normalizing, the orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ is the unique one (up to $\pm$ signs) with the property that $\operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_k\} = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ for all $k = 1, 2, \ldots$.

Example 70. Using Gram–Schmidt, find an orthogonal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Solution. We already have the basis $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ for W. However, that basis is not orthogonal.

We can construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for W as follows:

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_2 \text{ onto } \boldsymbol{q}_1 \end{array} \right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$

  Note. $\boldsymbol{q}_2$ is the error of the projection of $\boldsymbol{w}_2$ onto $\boldsymbol{q}_1$. This guarantees that it is orthogonal to $\boldsymbol{q}_1$. On the other hand, since $\boldsymbol{q}_2$ is a combination of $\boldsymbol{w}_2$ and $\boldsymbol{q}_1$, we know that $\boldsymbol{q}_2$ actually is in W.

We have thus found the orthogonal basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ for W (if we like, we can, of course, drop that $\frac{2}{3}$).

Important comment. By normalizing, we get an orthonormal basis for W: $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Practical comment. When implementing Gram–Schmidt on a computer, it is beneficial (slightly less work) to normalize each $\boldsymbol{q}_i$ during the Gram–Schmidt process. This typically introduces square roots, which is why normalizing at the end is usually preferable when working by hand.

Comment. There are, of course, many orthogonal bases $\boldsymbol{q}_1, \boldsymbol{q}_2$ for W. Up to the length of the vectors, ours is the unique one with the property that $\operatorname{span}\{\boldsymbol{q}_1\} = \operatorname{span}\{\boldsymbol{w}_1\}$ and $\operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\} = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$.

Example 71. Using Gram–Schmidt, find an orthogonal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution. We begin with the (not orthogonal) basis $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

We then construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$:

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_2 \text{ onto } \boldsymbol{q}_1 \end{array} \right) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $\boldsymbol{q}_3 = \boldsymbol{w}_3 - \left(\begin{array}{c} \text{projection of } \boldsymbol{w}_3 \\ \text{onto span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\} \end{array} \right) = \boldsymbol{w}_3 - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_1 \end{array} \right) - \left(\begin{array}{c} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_2 \end{array} \right)$

  $= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

  Make sure you understand how $\boldsymbol{q}_3$ was designed to be orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$!

  Also note that breaking up the projection onto $\operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ into the projections onto $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ is only possible because $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are orthogonal.

Hence, $\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is an orthogonal basis of W.

Important. Normalizing, we obtain an orthonormal basis: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

A matrix Q has orthonormal columns $\iff$ $Q^T Q = I$

Why? Let $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ be the columns of Q. By the way matrix multiplication works, the entries of $Q^T Q$ are dot products of these columns:

$$\begin{bmatrix} — & \boldsymbol{q}_1^T & — \\ — & \boldsymbol{q}_2^T & — \\ & \vdots & \end{bmatrix} \begin{bmatrix} | & | & \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots \\ | & | & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

Hence, $Q^T Q = I$ if and only if $\boldsymbol{q}_i^T \boldsymbol{q}_j = 0$ (that is, the columns are orthogonal), for $i \neq j$, and $\boldsymbol{q}_i^T \boldsymbol{q}_i = 1$ (that is, the columns are normalized).

Example 72. $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$ obtained from Example 70 satisfies $Q^T Q = I$.

The QR decomposition

Just like the LU decomposition encodes the steps of Gaussian elimination, the QR decomposition encodes the steps of Gram–Schmidt.

(QR decomposition) Every $m \times n$ matrix A of rank n can be decomposed as $A = QR$, where

- Q has orthonormal columns, $(m \times n)$
- R is upper triangular and invertible. $(n \times n)$

How to find Q and R?

- Gram–Schmidt orthonormalization on (columns of) A, to get (columns of) Q
- $R = Q^T A$
  Why? If $A = QR$, then $Q^T A = Q^T Q R$ which simplifies to $R = Q^T A$ (since $Q^T Q = I$).

The decomposition $A = QR$ is unique if we require the diagonal entries of R to be positive (and this is exactly what happens when applying Gram–Schmidt).

Practical comment. Actually, no extra work is needed for computing R. All of its entries have been computed during Gram–Schmidt.

Variations. We can also arrange things so that Q is an $m \times m$ **orthogonal** matrix (this means Q is square and has orthonormal columns) and R a $m \times n$ upper triangular matrix. This is a tiny bit more work (and not required for many applications): we need to complement "our" Q with additional orthonormal columns and add corresponding zero rows to R. For square matrices this makes no difference.

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Example 73. Determine the QR decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Solution. The first step is Gram–Schmidt orthonormalization on the columns of A. We then use the resulting orthonormal vectors (they need to be normalized!) as the columns of Q.

We already did Gram–Schmidt in Example 70: from that work, we have $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & 4/\sqrt{6} \end{bmatrix}$.

Comment. The entries of R have actually all been computed during Gram–Schmidt, so that, if we pay attention, we could immediately write down R (no extra work required). Looking back at Example 70, can you see this?

Check. Indeed, $QR = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & 4/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ equals A.

Example 74. Determine the QR decomposition of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. The first step is Gram–Schmidt orthonormalization on the columns of A. We then use the resulting orthonormal vectors as the columns of Q.

We already did Gram–Schmidt in Example 71: from that work, we have $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

Comment. As commented earlier, the entries of R have actually all been computed during Gram–Schmidt, so that, if we pay attention, we could immediately write down R (no extra work required). Looking back at Example 71, can you see this?

Letting Sage do the work for us.

```
Sage] A = matrix(QQbar, [[0,2,1],[3,1,1],[0,0,1],[0,0,1]])
```

```
Sage] A.QR(full=false)
```

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0.7071067811865475? \\ 0 & 0 & 0.7071067811865475? \end{bmatrix}, \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1.414213562373095? \end{bmatrix} \right)$$

Comment. Can you figure out what happens if you omit the `full=false`? Check out the comment under **Variations** in the statement of the QR decomposition. On the other hand, the `QQbar` is telling Sage to compute with algebraic numbers (instead of just rational numbers); if omitted, it would complain that square roots are not available

Example 75. (extra) Determine the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution. We first apply Gram–Schmidt orthonormalization to the columns of A. For a variation, like a computer, we normalize after each step (rather than normalize at the end):

- $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
- $\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, so that $\boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- $\boldsymbol{b}_3 = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 - \left(\begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} \cdot \boldsymbol{q}_2 \right) \boldsymbol{q}_2 = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Therefore, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. Finally, $R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

In conclusion, we have found the QR decomposition: $\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}}_{R}$

Comment. As noted before, we actually could write down R without any additional computation. Indeed, realize that the second column of R, that is $[2,3,0]^T$ means that

$$\text{2nd col of } A = 2\boldsymbol{q}_1 + 3\boldsymbol{q}_2.$$

Which we already knew from our computation of $\boldsymbol{q}_2$! Also, by construction, we know that the second column of A is a linear combination of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ only, and that $\boldsymbol{q}_3$ enters the story later on. This corresponds to the fact that R is always upper triangular.

Letting Sage do the work for us.

```
Sage] A = matrix(QQbar, [[1,2,4], [0,0,-5], [0,3,6]])
Sage] A.QR()
```

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \right)$$

Comment. The `QQbar` is telling Sage to compute with algebraic numbers (instead of just rational numbers); in general, if omitted, it would complain that square roots are not available (because the matrices Q and R typically involve square roots). Here, we are lucky that square roots didn't creep in.

Example 76. (extra) Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution. (final answer only) $A = QR$ with $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ and $R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$.

Example 77. One practical application of the QR decomposition is solving systems of linear equations.

$$\begin{aligned} A\boldsymbol{x} = \boldsymbol{b} \quad &\Longleftrightarrow \quad QR\boldsymbol{x} = \boldsymbol{b} \qquad \text{(now, multiply with } Q^T \text{ from the left)} \\ &\Longrightarrow \quad R\boldsymbol{x} = Q^T\boldsymbol{b} \end{aligned}$$

The last system is triangular and can be solved by back-substitution.

A couple of comments are in order:

- If A is $n \times n$ and invertible, then the "$\Longrightarrow$" is actually a "$\Longleftrightarrow$".
- The equation $R\boldsymbol{x} = Q^T\boldsymbol{b}$ is always consistent! (Recall that R is invertible.)
  Indeed, if A is not $n \times n$ or not invertible, then $R\boldsymbol{x} = Q^T\boldsymbol{b}$ gives the least squares solutions!
  Why? $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b} \quad \Longleftrightarrow \quad \underbrace{(QR)^TQR}_{=R^TQ^TQR}\hat{\boldsymbol{x}} = (QR)^T\boldsymbol{b} \quad \Longleftrightarrow \quad R^TR\hat{\boldsymbol{x}} = R^TQ^T\boldsymbol{b} \quad \Longleftrightarrow \quad R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$
  [For the last step we need that R is invertible, which is always the case when A is $m \times n$ of rank n.]
- So, how does the QR way of solving linear systems compare to our beloved Gaussian elimination (LU)?
  It turns out that QR is a little slower than LU but makes up for it in "numerical stability".
  What does that mean? When computing numerically, we use floating point arithmetic and approximate each number by an expression of the form $0.1234 \cdot 10^{-16}$. A certain (fixed) number of bits is used to store the part 0.1234 (here, 4 decimal places of accuracy) as well as the exponent -16.
  Now, here is something terrible that can happen in numerical computations: mathematically, the quantities x and $(x+1)-1$ are exactly the same. However, numerically, they might not. Take, for instance, $x = 0.1234 \cdot 10^{-6}$. Then, to an accuracy of 4 decimal places, $x + 1 = 0.1000 \cdot 10^1$, so that $(x+1) - 1 = 0.0000$. But $x \neq 0$. We completely lost all the information about x.
  To be numerically stable, an algorithm must avoid issues like that.

$\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$
$\Longleftrightarrow \quad R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$ (where $A = QR$)

Example 78. (review) In Example 17, we diagonalized $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ as $A = PDP^{-1}$.

We found that one choice for P and D is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Spell out what that tells us about A!

Solution. The diagonal entries $5, 2, 2$ of D are the eigenvalues of A.
The columns of P are corresponding eigenvectors of A.

- $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is a 5-eigenvector of A (that is, $A\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$).
- The 2-eigenspace of A is 2-dimensional. A basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Example 79. Diagonalize the **symmetric matrix** $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ as $A = PDP^{-1}$.

Review. Recall that a matrix A is symmetric if $A^T = A$.

Solution. We let Sage do the work for us:

```
Sage] A = matrix([[8,-6,2],[-6,7,-4],[2,-4,3]])
Sage] A.eigenmatrix_right()
```

$$\left(\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ -1 & \frac{1}{2} & 2 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} \right)$$

This ouput shows that A is diagonalizable as $A = PDP^{-1}$ with $D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & \frac{1}{2} & 2 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$.

Just to make sure. This means that the eigenvalues of A are $15, 3, 0$ (the diagonal entries of D).

Moreover, we have that $\begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$ is a 15-eigenvector, $\begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ is a 3-eigenvector, and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a 0-eigenvector.

Important observation. Note that the eigenspaces of A are orthogonal to each other here.
The **spectral theorem** says that this is true for all symmetric matrices A.

Example 80. Diagonalize the symmetric matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ as $A = PDP^T$.

Solution. By the previous example, we can diagonalize A as $\tilde{P}D\tilde{P}^{-1}$ with $\tilde{P} = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(To avoid fractions, we just scaled the first two columns of $\tilde{P}$, which are eigenvectors.)
Note that the columns of $\tilde{P}$ are orthogonal (this is due the spectral theorem). If we normalize them (they all have length $\sqrt{2^2 + 2^2 + 1} = 3$), then we obtain the orthogonal matrix $P = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$.
Since $P^{-1} = P^T$, we now have $A = PDP^T$.

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Example 81.

(a) Determine the eigenspaces of the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(b) Diagonalize A as $A = PDP^T$.

Solution.

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$, and so A has eigenvalues $5, 0$.

The 5-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The 0-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Important observation. The 5-eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the 0-eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal!

(b) Note that a usual diagonalization is of the form $A = PDP^{-1}$.

We need to choose P so that $P^{-1} = P^T$, which means that P must be **orthogonal** (meaning orthonormal columns). [Choosing such a P is only possible if the eigenspaces of A are orthogonal.]

Hence, we normalize the two eigenvectors to $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

With $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, we then have $A = PDP^T$.

Review. A matrix A has orthonormal columns $\iff$ $A^TA = I$.

Example 82. Suppose Q has orthonormal columns. What is the projection matrix P for orthogonally projecting onto $\text{col}(Q)$?

Solution. Recall that, to project onto $\text{col}(A)$, the projection matrix is $P = A(A^TA)^{-1}A^T$.

Since $Q^TQ = I$, to project onto $\text{col}(Q)$, the projection matrix is $P = QQ^T$.

Comment. A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q}\cdot\boldsymbol{b})\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}^T\boldsymbol{b}) = (\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.

Comment. In particular, if Q is not square, then $Q^TQ = I$ but $QQ^T \neq I$. In some sense, QQ^T still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\text{col}(Q)$ it does act like the identity for vectors in $\text{col}(Q)$. (Vectors not in $\text{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\text{col}(Q)$.)

Example 83. Suppose A is invertible. What is the projection matrix P for orthogonally projecting onto $\text{col}(A)$?

Solution. If A is an invertible $n \times n$ matrix, then $\text{col}(A) = \mathbb{R}^n$ (because the n columns of A are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\text{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

Definition 84. An **orthogonal matrix** is a square matrix with orthonormal columns.

[This is not a typo (but a confusing convention): the columns need to be orthonormal, not just orthogonal.]

An $n \times n$ matrix Q is orthogonal $\iff$ $Q^TQ = I$

In other words, $Q^{-1} = Q^T$.

Review. Recall the following properties of determinants:

- $\det(AB) = \det(A)\det(B)$

  Comment. In fancy language, this means that the determinant is a group homomorphism between the group of (invertible) $n \times n$ matrices and (nonzero) complex numbers. Note that, on the left hand, we are multiplying matrices while, on the right hand, we are multiplying numbers. The key point is that it doesn't matter which multiplication we do: the two multiplications are compatible.

- $\det(A^{-1}) = \frac{1}{\det(A)}$

  Comment. Can you derive this from the previous property?

- $\det(A^T) = \det(A)$

  Comment. We are familiar with this in the context of cofactor expansion: it doesn't matter whether we expand by a column or by a row.

Example 85. What can we say about $\det(Q)$ if Q is orthogonal?

Solution. Write $d = \det(Q)$. Since $Q^{-1} = Q^T$, we have $\frac{1}{d} = d$ (recall that $\det(Q^{-1}) = 1/\det(Q)$ and $\det(Q^T) = \det(Q)$) or, equivalently, $d^2 = 1$. Hence, $d = \pm 1$.

Both of these are possible as the examples $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ illustrate.

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Review: Diagonalizability

Example 86. (review) If A is a 2×2 matrix with $\det(A) = -8$ and eigenvalue 4. What is the second eigenvalue?

Solution. Recall that $\det(A)$ is the product of the eigenvalues (see below). Hence, the second eigenvalue is -2.

$\det(A)$ is the product of the eigenvalues of A.

Why? Recall how we determine the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A. We compute the characteristic polynomial $\det(A - \lambda I)$ and determine the λ_i as the roots of that polynomial.

That means that we have the factorization $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_n - \lambda) \cdots (\lambda_n - \lambda)$. Now, set $\lambda = 0$ to conclude that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Lemma 87. A matrix A is diagonalizable if and only if, for every eigenvalue λ that is k times repeated, the λ-eigenspace of A has dimension k.

In short, an $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e. "there are enough eigenvectors").

The next two examples illustrate that not all matrices are diagonalizable and that, even if a real matrix is diagonalizable, the eigenvalues and eigenvectors might be complex.

Example 88. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2$, which has $\lambda = 0$ as a double root.

However, the 0-eigenspace $\operatorname{null}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is only 1-dimensional.

As a consequence, A is not diagonalizable.

Example 89. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$.

Hence, the eigenvalues are $\pm i$.

The i-eigenspace $\operatorname{null}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right)$ has basis $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

The $-i$-eigenspace $\operatorname{null}\left(\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\right)$ has basis $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Thus, A has the diagonalization $A = PDP^{-1}$ with $D = \begin{bmatrix} i & \\ & -i \end{bmatrix}$ and $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$.

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The spectral theorem

Recall that a matrix A is symmetric if and only if $A^T = A$.

Theorem 90. (spectral theorem, long version) Suppose A is a symmetric matrix.

- A is always diagonalizable.
- All eigenvalues of A are real.
- The eigenspaces of A are orthogonal.

Proof. We will prove (parts of) the spectral theorem later on. For now, we just appreciate that the spectral theorem guarantees all these nice things to happen for symmetric matrices (for any specific A we know how to determine whether A is diagonalizable and what its eigenspaces are).

Comment. The eigenspaces of A being orthogonal means that eigenvectors for different eigenvalues are always orthogonal.

Important consequence. In the diagonalization $A = PDP^{-1}$, we can choose P to be orthogonal (in which case $P^{-1} = P^T$). In that case, the diagonalization takes the special form $A = PDP^T$, where P is orthogonal and D is diagonal.

(spectral theorem, compact version) A symmetric matrix A can always be diagonalized as $A = PDP^T$, where P is orthogonal and D is diagonal (and both are real).

How? We proceed as in the diagonalization $A = PDP^{-1}$. For a symmetric matrix A, we can arrange P to be orthogonal, by normalizing its columns. If there is a repeated eigenvalue, then we also need to make sure to pick an orthonormal basis for the corresponding eigenspace (for instance, using Gram–Schmidt).

Advanced comment. A matrix such that $A^T A = A A^T$ is called **normal**. For normal matrices, the (complex!) eigenspaces are again orthogonal to each other. However, normal matrices which are not symmetric will always have complex eigenvalues. (In that case, the orthogonal matrix P gets replaced with a unitary matrix, the complex version of orthogonal matrices, and the P^T becomes the conjugate transpose $P^* = \bar{P}^T$.)

Example 91.

(a) Determine the eigenspaces of the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(b) Diagonalize A as $A = PDP^T$.

Solution.

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2)$, and so A has eigenvalues $4, -2$.

The 4-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The -2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Important observation. The 4-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the -2-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthogonal!

Review. The product of all eigenvalues $-2 \cdot 4 = -8$ equals the determinant $\det(A) = 1 - 9 = -8$.

(b) Note that a usual diagonalization is of the form $A = PDP^{-1}$.

We need to choose P so that $P^{-1} = P^T$, which means that P must be **orthogonal** (meaning orthonormal columns). [Choosing such a P is only possible if the eigenspaces of A are orthogonal.]

Hence, we normalize the two eigenvectors to $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

With $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$, we then have $A = PDP^T$.

Example 92. (again, simplified) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ as $A = PDP^T$.

Solution. See Example 91 for a solution that illustrates how to diagonalize any symmetric matrix. For a simplified solution, note that we can see right away that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 4-eigenvector (since the row sums are equal!).

Because the eigenspaces are orthogonal (since A is symmetric!), $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ must also be an eigenvector.

Indeed, $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ shows that the corresponding eigenvalues is -2.

We normalize the two eigenvectors and use them as the columns of P, so that $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is an orthogonal matrix ($P^{-1} = P^T$). With $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ we then have $A = PDP^T$.

Example 93. Let A be a symmetric 2×2 matrix with 7-eigenvector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\det(A) = -21$. Determine the second eigenvalue and a corresponding eigenvector.

Further, diagonalize A as $A = PDP^T$.

Solution. A has $-\frac{21}{7} = -3$-eigenvector $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

Hence, $A = PDP^T$ with $D = \begin{bmatrix} 7 & \\ & -3 \end{bmatrix}$ and $P = \frac{1}{\sqrt{29}}\begin{bmatrix} 2 & -5 \\ 5 & 2 \end{bmatrix}$.

Comment. Recall that, because A is symmetric, the eigenvector must be orthogonal to $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

[In general, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} -b \\ a \end{bmatrix}$ are orthogonal.]

Let us prove the following important part of the spectral theorem.

Theorem 94. If A is symmetric, then the eigenspaces of A are orthogonal.

Proof. To prove the claim we need to show that, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues (say λ and μ), then $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{0}$. Suppose therefore that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and $A\boldsymbol{w} = \mu\boldsymbol{w}$ with $\lambda \neq \mu$.

First, we observe that, for any matrix A and vectors $\boldsymbol{v}$, $\boldsymbol{w}$, we have the following:

$$(A\boldsymbol{v}) \cdot \boldsymbol{w} = (A\boldsymbol{v})^T\boldsymbol{w} = (\boldsymbol{v}^T A^T)\boldsymbol{w} = \boldsymbol{v}^T(A^T\boldsymbol{w}) = \boldsymbol{v} \cdot (A^T\boldsymbol{w})$$

If A is symmetric, we therefore have

$$(A\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot (A\boldsymbol{w}).$$

We now use that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and $A\boldsymbol{w} = \mu\boldsymbol{w}$ to conclude from the above that $\lambda\boldsymbol{v} \cdot \boldsymbol{w} = \mu\boldsymbol{v} \cdot \boldsymbol{w}$.

However, since $\lambda \neq \mu$, this is only possible if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. □

Example 95. (review) Let A be a symmetric 2×2 matrix with 2-eigenvector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\det(A)=8$. Determine the second eigenvalue and a corresponding eigenvector.
Further, diagonalize A as $A=PDP^T$.

Solution. A has $8/2=4$-eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (chosen to be orthogonal to $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ since, by the spectral theorem, symmetric matrices have orthogonal eigenspaces).
Hence, $A=PDP^T$ with $D=\begin{bmatrix} 2 & \\ & 4 \end{bmatrix}$ and $P=\frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$.

Powers of matrices

Example 96. (warmup) Consider $A=\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

- What are the eigenspaces?
- What are A^{-1} and A^{100}? What is A^n?

Solution.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a -2-eigenvector, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 3-eigenvector. In other words, the -2-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and the 3-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.
- $A^{-1}=\begin{bmatrix} -1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ and $A^{100}=\begin{bmatrix} (-2)^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}=\begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$. In general, $A^n=\begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix}$.

Comment. Algebraically, the map $\boldsymbol{v}\mapsto A\boldsymbol{v}$ looks very simple. However, notice that it is not so easy to say what happens to, say, $\boldsymbol{v}=\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ geometrically. That is because two things are happening: part of the vector $\boldsymbol{v}$ is scaled by -2, the other part is scaled by 3.

Example 97. If A has λ-eigenvector $\boldsymbol{v}$, then what can we say about A^2?

Solution. A^2 has λ^2-eigenvector $\boldsymbol{v}$.
[Indeed, $A^2\boldsymbol{v}=A(A\boldsymbol{v})=A(\lambda\boldsymbol{v})=\lambda A\boldsymbol{v}=\lambda^2\boldsymbol{v}$. This is even easier in words: multiplying $\boldsymbol{v}$ with A has the effect of scaling it by λ; hence, multiplying it with A^2 scales it by λ^2.]
Important comment. Similarly, A^{100} has λ^{100}-eigenvector $\boldsymbol{v}$.

Example 98. If a matrix A can be diagonalized as $A=PDP^{-1}$, what can we say about A^n?

Solution. First, note that $A^2=(PDP^{-1})(PDP^{-1})=PD^2P^{-1}$. Likewise, $A^n=PD^nP^{-1}$.
[The point being that D^n is trivial to compute because D is diagonal.]
In particular. $A^{-1}=PD^{-1}P^{-1}$
Important comment. In the previous example, we observed that, if A has λ-eigenvector $\boldsymbol{v}$, then A^n has λ^n-eigenvector $\boldsymbol{v}$. Note that this is also expressed in $A^n=PD^nP^{-1}$, because the latter is a diagonalization of A^n. The diagonalization shows that A^n and A have the same eigenvectors (since we can use the same matrix P) and that the eigenvalues of A^n are the n-th powers of the eigenvalues of A (which are the entries of the diagonal matrix D).

(computing matrix powers) If A is a square matrix with diagonalization $A=PDP^{-1}$, then

$$A^n=PD^nP^{-1}.$$

Example 99. Let $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$. Compute A^n.

Solution. First, we diagonalize: $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 10 & \\ & 5 \end{bmatrix}$. (Fill in the details!)

$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 10^n & \\ & 5^n \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 10^n & 10^n \\ -4\cdot 5^n & 1\cdot 5^n \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 10^n + 4\cdot 5^n & 10^n - 5^n \\ 4\cdot 10^n - 4\cdot 5^n & 4\cdot 10^n + 5^n \end{bmatrix}$

Check. Verify the cases $n = 0$ ($A^0 = I$) and $n = 1$.

Example 100. (extra) Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$. Determine A^n.

Solution. We first repeat our work from Example 17 to find a diagonalization of A:
By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda) - 2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 5$.

- $\lambda = 5$: $\text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\text{RREF}}{=} \text{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$
- $\lambda = 2$: $\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\text{RREF}}{=} \text{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

We therefore have the diagonalization $A = PDP^{-1}$ with $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

[Keep in mind that other choices for P and D exist.]

With some labor (do it!), we find $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix}$.

It follows that

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2\cdot 5^n & 0 & -2^n \\ 2\cdot 5^n & 2^n & 0 \\ 5^n & 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2\cdot 5^n + 2^n & 0 & 2\cdot 5^n - 2\cdot 2^n \\ 2\cdot 5^n - 2\cdot 2^n & 3\cdot 2^n & 2\cdot 5^n - 2\cdot 2^n \\ 5^n - 2^n & 0 & 5^n + 2\cdot 2^n \end{bmatrix}. \end{aligned}$$

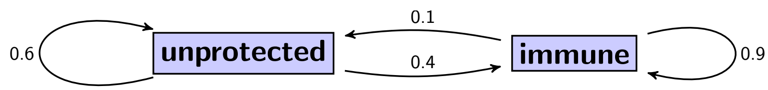
Check. Notice that it is particularly easy to verify the cases $n = 0$ ($A^0 = I$) and $n = 1$.

Application: Markov chains

Example 101. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.

What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)

y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

Powers of the transition matrix. Note that $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. In other words, M^n describes the transition over n years. In particular, the powers of M are the key to understanding what happens in this model over time.

Equilibrium. $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$. In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{1+4}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$.

Hence, the immunization rate in the long term equilibrium is $4/5 = 80\%$.

[Ponder about why this is a reasonable value!]

Comment. What's the other eigenvalue of the transition matrix? No need to compute the characteristic polynomial: we can easily see that it is $0.5 = 0.6 \cdot 0.9 - 0.1 \cdot 0.4$ because the product of the eigenvalues equals the determinant!

The 0.5-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Comment. Will the immunization rate always stabilize and approach the long term equilibrium? Yes! This is a consequence of the other eigenvalue of the transition matrix satisfying $|0.5| < 1$. If we start in state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 4 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1^n \cdot a\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (0.5)^n \cdot b\begin{bmatrix} -1 \\ 1 \end{bmatrix} \underset{\text{as } n \to \infty}{\longrightarrow} a\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Random comment. A rule of thumb is that a tetanus vaccination begins to wear off after about 10 years (somewhat in line with the 0.1 transition proportion in this example). However, the tetanus immunzation rate in the United States appears to be considerable less than the 80% we found in this (awfully simplistic) example.

https://www.cdc.gov/mmwr/preview/mmwrhtml/mm5940a3.htm

Example 102. Compute M^n for $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$.

Solution. In Example 99, we computed that $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$ had powers $A^n = \frac{1}{5}\begin{bmatrix} 10^n + 4 \cdot 5^n & 10^n - 5^n \\ 4 \cdot 10^n - 4 \cdot 5^n & 4 \cdot 10^n + 5^n \end{bmatrix}$.

Since $M = \frac{1}{10}A$, this implies that $M^n = \frac{1}{10^n}A^n = \frac{1}{5}\begin{bmatrix} 1 + 4 \cdot 0.5^n & 1 - 0.5^n \\ 4 - 4 \cdot 0.5^n & 4 + 0.5^n \end{bmatrix}$.

Note that $M^n \to \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$ as $n \to \infty$. This reflects the fact that $\frac{1}{5}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the long term equilibrium.

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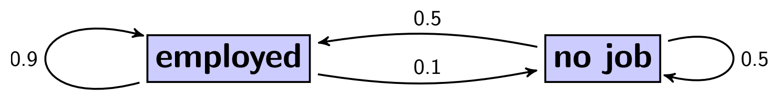
Example 103. (extra) Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Solution. Let x_t and y_t be the proportions of those employed and unemployed. Proceeding, as in the previous example, the transition matrix is $M = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$.



The 1-eigenspace of M, that is $\operatorname{null}\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix}\right)$, has basis $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$. The corresponding equilibrium is $\frac{1}{5+1}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

In particular, the unemployment rate in the long term equilibrium is $1/6 \approx 16.7\%$.

Example 104. Which of the following are true for all square matrices A?

- Is it true that A^T has the same eigenvalues as A?
- Is it true that A^T has the same eigenspaces as A?
- Is it true that A^T has the same characteristic polynomial as A?

Solution. True. False. True.

First, note that the characteristic polynomial $\det(A - \lambda I)$ is the same as $\det(A^T - \lambda I)$. [Make sure you can fill in the details of why this is the case!] Hence, the eigenvalues (which are the roots of the characteristic polynomial) are also the same for A and A^T.

On the other hand, A^T and A in general have very different eigenspaces. Take, for instance, the matrix $A = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ from Example 101. Then both A and A^T have eigenvalues $\lambda = 0.5, 1$.

However, the 1-eigenspace of A is spanned by $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, while the 1-eigenspace of A^T is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 105. Show that a Markov matrix A (so that the columns of A sum to 1) always has eigenvalue 1.

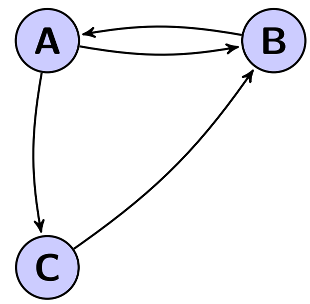
Solution. This follows because the transpose A^T always has $\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$ as an 1-eigenvector (by virtue of the rows of A^T summing to 1). [Make sure that makes sense!]

By the previous example, A must also have eigenvalue 1 (but we have no idea what a 1-eigenvector is until we compute it).

Application: PageRank

Example 106. Suppose the internet consists of only the three webpages A, B, C.

We wish to rank these webpages in order of "importance".



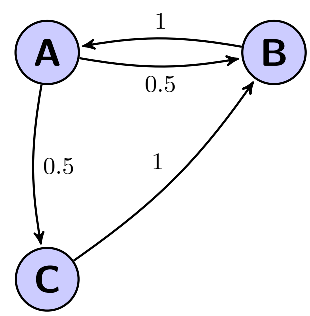
The idea. Instead of analyzing each webpage (which would be a lot of work!) we will try to only use the information how the pages are linked to each other. The idea being that an "important" page should be linked to from many other pages.

A and B have a link to each other. Also, A links to C and C links to B. If you keep randomly clicking from one webpage to the next, what proportion of the time will you be at each page?

The idea. We will assign ranking to the pages according to how frequently such a random surfer would visit these pages.

Comment. Before we start computing, stop for a moment, and think about how you would rank the webpages.

Solution. Let a_t be the probability that we will be on page A at time t. Likewise, b_t, c_t are the probabilities that we will be on page B or C.



The transition from one state to the next now works exactly as in the previous example. We get the following transition matrix:

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + 1 \cdot b_t + 0 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 1 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

To find the equilibrium state, we again determine an appropriate 1-eigenvector.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix}\right)$ which has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. In this context, this is also known as the **PageRank vector**.

In other words, after browsing randomly for a long time, there is (about) a $\frac{2}{5} = 40\%$ chance to be at page A, a $\frac{2}{5} = 40\%$ chance to be at page B, and a $\frac{1}{5} = 20\%$ chance to be at page C.

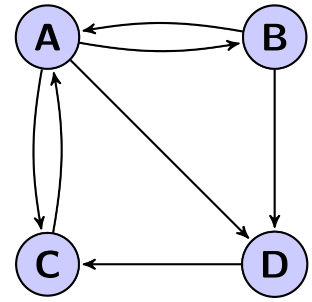
We therefore rank A and B highest (tied), and C lowest.

Just checking. Maybe we were expecting B to be ranked above A, because B is the only page that has two incoming links. However, if we are at page B, then our next click will be to page A, which is why A and B receive equal ranking.

This method of ranking is the famous **PageRank** algorithm (underlying Google's search algorithm).

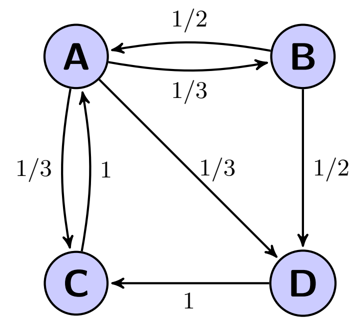
By the way, the algorithm is named, not after ranking web"pages", but after Larry Page (who founded Google in 1998 together with Sergey Brin).

Example 107. Suppose the internet consists of only the four webpages A, B, C, D which link to each other as indicated in the diagram. Rank these webpages by computing their PageRank vector.



Solution. Recall that we model a random surfer, who randomly clicks on links. Let a_t be the probability that such a surfer will be on page A at time t. Likewise, b_t, c_t, d_t are the probabilities that the surfer will be on page B, C or D.

The transition probabilities are indicated in the diagram to the right. As in the previous example, we obtain the following transition behaviour:



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + \frac{1}{2} \cdot b_t + 1 \cdot c_t + 0 \cdot d_t \\ \frac{1}{3} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ \frac{1}{3} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 1 \cdot d_t \\ \frac{1}{3} \cdot a_t + \frac{1}{2} \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}}_{=M} \begin{bmatrix} a_t \\ b_t \\ c_t \\ d_t \end{bmatrix}$$

To find the equilibrium state, we determine an appropriate 1-eigenvector of the transition matrix M.

The 1-eigenspace is $\operatorname{null}(M - 1 \cdot I) = \operatorname{null}\left(\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix}\right)$.

To compute a basis, we perform Gaussian elimination: $\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We conclude that the 1-eigenspace has basis $\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$. (Note that its entries add up to $2 + \frac{2}{3} + \frac{5}{3} + 1 = \frac{16}{3}$.)

The corresponding equilibrium state is $\frac{3}{16} \begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$. This is the **PageRank vector**.

[For instance, after browsing randomly for a long time, there is (about) a 12.5% chance to be at page B.]
Correspondingly, we rank the pages as $A > C > D > B$.

The real internet. [Google is getting more secretive about this kind of data, so the numbers are estimates from a while ago.]

- Google reports (2016) doing "trillions" of searches per year. [2 trillion means 63,000 searches per second.]
- Google's search index contains almost 50 billion pages (2016). [Estimated to exceed 100,000,000 gigabytes.]
- More than 1,000,000,000 websites (i.e. hostnames; about 75% not active)

[The "average" user apparently only visits about 100 websites per month; wikipedia.org is one website, consisting of many webpages (more than 2,000,000).]

Gory details. (extra) There's nothing interesting about the Gaussian elimination above. Here are the full details:

$$\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_2+\frac{1}{3}R_1 \Rightarrow R_2 \\ R_3+\frac{1}{3}R_1 \Rightarrow R_3 \\ R_4+\frac{1}{3}R_1 \Rightarrow R_4}} \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & -\frac{2}{3} & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_3+\frac{1}{5}R_2 \Rightarrow R_3 \\ R_4+\frac{4}{5}R_2 \Rightarrow R_4}} \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{3}{5} & 1 \\ 0 & 0 & \frac{3}{5} & -1 \end{bmatrix}$$

$$\underset{\rightsquigarrow}{R_4+R_3 \Rightarrow R_4} \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{3}{5} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{\substack{-1R_1 \Rightarrow R_1 \\ -\frac{6}{5}R_2 \Rightarrow R_2 \\ -\frac{5}{3}R_3 \Rightarrow R_3}} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_1+R_3 \Rightarrow R_1 \\ R_2+\frac{2}{5}R_3 \Rightarrow R_2}} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{R_1+\frac{1}{2}R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Practical comment. The transition matrix we would get for the entire internet indexed by Google is prohibitingly large (a 50 billion by 50 billion matrix). While gigantic in size, it is a very **sparse matrix**, meaning that almost all of its entries are zero (each column has 50 billion entries but only a handful are nonzero, namely those corresponding to a link to another webpage). This is typical for many applications in linear algebra: we often deal with big but sparse matrices.

Another practical comment. It's not an issue in our simple example, but what if our random surfer gets stuck on a webpage without links? Or, similarly, gets stuck in a loop of links? To deal with these, it is customary to include "teleportation". That is, each time, one of two things happens: with probability p (typically, something like $p = 0.85$) our surfer clicks a link as before; otherwise, with probability $1 - p$, he is teleported to some unrelated other page. Further, if the surfer comes to a page without links, he would teleport away.

A final practical comment. In practical situations, the system might be too large for finding the equilibrium vector by elimination, as we did above. An alternative to elimination is the power method: it is based on the idea that the equilibrium vector is what we expect in the long-term. We can approximate this "long-term" behaviour by simulating a few transitions. For instance, in our example, if we start with the state $[\ 1/4\ \ 1/4\ \ 1/4\ \ 1/4\ ]^T$, which corresponds to equal chances of being on each webpage, then the next state (that is, after one random click) is

$$M \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/12 \\ 1/3 \\ 5/24 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}.$$

Note that the ranking of the webpages is already A, C, D, B if we stop right here.

The state after that (that is, after two random clicks) is $M^2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.333 \\ 0.167 \end{bmatrix}$, and $M^3 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.396 \\ 0.125 \\ 0.292 \\ 0.188 \end{bmatrix}$.

Observe how we are (overall) approaching the equilibrium vector $\begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$.

Iterating like this is guaranteed to converge to a 1-eigenvector under mild technical assumptions on the transition matrix (for instance, that all its entries be positive; in that case, the other eigenvalues λ satisfy $|\lambda| < 1$ so that their contributions go to zero exponentially, as in Example 101).

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Application: Fibonacci numbers

The numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

How fast are they growing? Have a look at ratios of Fibonacci numbers:
$\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.667$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} \approx 1.615$, $\frac{34}{21} \approx 1.619$, ...

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

In other words, it appears that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 115 below.

The crucial insight is the following simple observation:

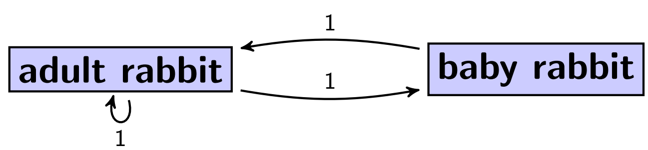
$$F_{n+2} = F_{n+1} + F_n \quad \text{is equivalent to} \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

In particular, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

Comment. Recurrence equations are discrete analogs of differential equations. We will later see the same idea applied when we reduce the order of a differential equation by introducing additional equations.

Example 108. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these features might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let a_t be the number of adult rabbit pairs after t months. Likewise, b_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} a_t + b_t \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix}.$$

That's precisely the transition for the Fibonacci numbers!

It follows that Fibonacci numbers count the number of rabbits in this model.

Comment. Note that the setup is very much as for Markov chains. Here, however, the outgoing values do not add to 100% for each state. Consequently, we cannot expect an equilibrium (and, indeed, the number of rabbits increases without bound).

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Note that, given any Fibonacci-like recursion, we can apply our linear algebra skills in the same fashion. The next example illustrates how this is set up.

Example 109. Suppose the sequence a_n satisfies $a_{n+3} = 3a_{n+2} - 2a_{n+1} + 7a_n$. Write down a matrix-vector version of this recursion.

Solution. $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$

Important. If we write $\boldsymbol{a}_n = \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$, then this is simply $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $M = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

In particular, it follows that $\boldsymbol{a}_n = M^n \boldsymbol{a}_0$.

If we compute M^n, then this produces an explicit formula for a_n (the third entry of $\boldsymbol{a}_n$). This formula is called a **Binet-like formula** (in the case of the Fibonacci numbers, this is precisely the classical Binet formula).

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Example 110. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -1$, $a_1 = 5$.

(a) Determine the first few terms of the sequence.

(b) Write down a matrix-vector version of the recursion.

(c) Find an explicit formula for a_n.

(d) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $-1, 5, 7, 29, 79, 245, 727, 2189, 6559, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

(c) **(solution using matrix powers)** Thus, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.

After some work (do it!), we diagonalize $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$ with $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix}}_{= \begin{bmatrix} 3^{n+1} & (-1)^{n+1} \\ 3^n & (-1)^n \end{bmatrix}} \underbrace{\frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{= \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 3^{n+1} - 2(-1)^{n+1} \\ 3^n - 2(-1)^n \end{bmatrix}$.

In particular, $a_n = 3^n - 2(-1)^n$.

(simplified solution) The eigenvalues of $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ are 3 and -1.

Looking back at our work above, we can see that a_n therefore must have a formula of the form $a_n = C_1 \cdot 3^n + C_2 \cdot (-1)^n$ for some unknown constants C_1, C_2 which we still need to figure out

Using the two initial conditions, we get two equations:

($a_0 =$) $C_1 + C_2 = -1$, ($a_1 =$) $3C_1 - C_2 = 5$.

Solving, we find $C_1 = 1$ and $C_2 = -2$ so that, in conclusion, $a_n = 3^n - 2(-1)^n$.

(d) It follows from the explicit formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_1 = 0$.

To be very precise. To see that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ we can observe that (by dividing each term by 3^n)

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1} - 2(-1)^{n+1}}{3^n - 2(-1)^n} = \frac{3 + 2\left(-\frac{1}{3}\right)^n}{1 - 2\left(-\frac{1}{3}\right)^n} \xrightarrow{\text{as } n\to\infty} \frac{3+0}{1-0} = 3.$$

Definition 111. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = r_1 a_{n+d-1} + r_2 a_{n+d-2} + \ldots + r_d a_n$$

is called ***C*-finite** (or, **constant recursive**) of order d.

For instance. For the Fibonacci numbers, $d = 2$ and $r_1 = r_2 = 1$.

In matrix-vector form. $\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 & r_2 & \cdots & r_{d-1} & r_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$

By the same reasoning as in the previous example, C-finite sequences always have and explicit formula (called Binet-like because it is known as the Binet formula for the Fibonacci numbers):

Theorem 112. (generalized Binet formula) Suppose the recursion matrix M has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + \ldots + C_d \lambda_d^n$$

for certain numbers $C_1, \ldots, C_d$. If, in addition, the eigenvalue λ_1 is larger in absolute value than the others (i.e. $|\lambda_1| > |\lambda_j|$ for $j = 2, 3, \ldots, d$) and $C_1 \neq 0$, then

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda_1.$$

How to see the limit. The limit is a consequence of $a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + \ldots + C_d \lambda_d^n$ because, for large n, the term $C_1 \lambda_1^n$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{C_1 \lambda_1^{n+1} + C_2 \lambda_2^{n+1} + \ldots + C_d \lambda_d^{n+1}}{C_1 \lambda_1^n + C_2 \lambda_2^n + \ldots + C_d \lambda_d^n} = \frac{C_1 \lambda_1 + C_2 \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d \lambda_d \left(\frac{\lambda_d}{\lambda_1}\right)^n}{C_1 + C_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d \left(\frac{\lambda_d}{\lambda_1}\right)^n} \overset{n \to \infty}{\longrightarrow} \frac{C_1 \lambda_1}{C_1} = \lambda_1.$$

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1 + \sqrt{5}}{2}$, $\lambda_2 = \frac{1 - \sqrt{5}}{2}$, and $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$. See Theorem 115.

Comment. More care is needed in the case that eigenvalues are repeated. Also, we need to be careful if there are several roots of the same absolute value. Consider, for instance, the case $a_n = 2^n + (-2)^n$. Can you see that, in this case, the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ doesn't even exist?

Example 113. Consider the sequence a_n defined by $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ and $a_0 = 0$, $a_1 = -2$, $a_2 = 2$.

(a) Determine the first few terms of the sequence.

(b) Find an explicit (Binet-like) formula for a_n.

(c) Determine $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, -2, 2, 10, 50, 178, 602, 1930, 6050, \ldots$
Note that this sequence is C-finite of order 3.

(b) The recursion can be translated to $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

Expanding by the 2nd row: $\begin{vmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -6 \\ 1 & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 4-\lambda & -6 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6$

The eigenvalues of the transition matrix are the roots of this polynomial: $\lambda = -1, 2, 3$

[You will not be asked to find roots of cubic polynomials by hand.]

Hence, $a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot 3^n$ and we only need to figure out the two unknowns C_1, C_2, C_3.

Using the three initial conditions, we get three equations:

(a_0=) $C_1 + C_2 + C_3 = 0$, ($a_1 =$) $-C_1 + 2C_2 + 3C_3 = -2$, ($a_2 =$) $C_1 + 4C_2 + 9C_3 = 2$.

Solving, we find $C_1 = 1$, $C_2 = -2$ and $C_3 = 1$ so that, in conclusion, $a_n = (-1)^n - 2 \cdot 2^n + 3^n$.

Comment. Do you see how we might have found the characteristic polynomial directly from the recursion?

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_3 = 0$.

Example 114. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{5}$. Hence, $a_n = C_1(1+\sqrt{5})^n + C_2(1-\sqrt{5})^n$ for certain numbers C_1, C_2.

[Note that we cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{5})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Let us work out an explicit formula for the Fibonacci numbers. This works exactly as in our previous examples, except that the eigenvalues involve square roots.

Theorem 115. (Binet's formula) $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$

Proof.

- We already observed that thee recurrence $F_{n+2} = F_{n+1} + F_n$ translates into $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $M = PDP^{-1}$ with

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

  Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = M^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} &= PD^nP^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix} \end{aligned}$$

- Hence, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, which is the claimed formula. □

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$.

Back to the quotients of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Comment. It follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Review. Recurrence equations, diagonalization, explicit (Binet-like) formulas

Example 116. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 5a_n$ and $a_0 = 0$, $a_1 = 1$.

(a) Determine the first few terms of the sequence.

(b) Find an explicit (Binet-like) formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, 1, 2, 9, 28, 101, 342, 1189, 4088, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{6}$.

Hence, $a_n = C_1(1+\sqrt{6})^n + C_2(1-\sqrt{6})^n$ and we only need to figure out the values of C_1 and C_2.
Using the two initial conditions, we get two equations:
($a_0 =$) $C_1 + C_2 = 0$, ($a_1 =$) $C_1(1+\sqrt{6}) + C_2(1-\sqrt{6}) = 1$.
Solving, we find $C_1 = \frac{1}{2\sqrt{6}}$ and $C_2 = -\frac{1}{2\sqrt{6}}$ so that, in conclusion, $a_n = \frac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$.

Comment. Alternatively, we could have proceeded as we did previously in the case of the Fibonacci numbers: starting with the recursion matrix M, we compute its diagonalization $M = PDP^{-1}$. Multiplying out $PD^nP^{-1}\begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{6} \approx 3.44949$.

Comment. Actually, we don't need the Binet-like formula for this conclusion. Just the eigenvalues and the observation that C_1 cannot be 0 are enough. [We cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{6})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Another brief look at projections (and reflections)

(projections) Suppose that M is the projection matrix for projecting onto a subspace W.

- The 1-eigenspace of M is W.
- The 0-eigenspace of M is $W^\perp$.

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get projected to themselves. But those are precisely the vectors in W (recall that projecting a vector $\boldsymbol{v}$ onto W means producing the vector in W that is closest to $\boldsymbol{v}$). Can you likewise spell out the situation for the 0-eigenspace?

Note that M cannot have further eigenvalues (because the dimensions of W and $W^\perp$ already add up to the dimension of the space that we are working in).

Because the eigenvalues of M are real and the eigenspaces are orthogonal, the matrix M has a diagonalization of the form $M = PDP^T$ (make sure you can explain why!) which implies that M is symmetric (that's because $M^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = M$).

Example 117. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^{-1}$.

(b) Diagonalize A as $A = PDP^T$.

Comment. This gives us yet another way to get our hands on projection matrices: we can directly write down the matrices P, D for the diagonalization $A = PDP^T$. The main point here is that the diagonalization of a A nicely reveals all the information about the projection.

[Can you see that this is not really a "new" way of computing projection matrices? In particular, note that, if Q is the matrix P with the third column omitted, then $A = QQ^T$.]

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional).

We already have a basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4 & 0 & -1/4 \\ 0 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (of the 1-eigenspace), we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{17}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{17}\begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$.

Next, we normalize the vectors $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{17}\begin{bmatrix} -4 \\ 34 \\ 16 \end{bmatrix}$, $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$ to $\frac{1}{\sqrt{17}}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{357}}\begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$, $\frac{1}{\sqrt{21}}\begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4/\sqrt{17} & -2/\sqrt{357} & -1/\sqrt{21} \\ 0 & 17/\sqrt{357} & -2/\sqrt{21} \\ 1/\sqrt{17} & 8/\sqrt{357} & 4/\sqrt{21} \end{bmatrix}$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{21}\begin{bmatrix} 20 & -2 & 4 \\ -2 & 17 & 8 \\ 4 & 8 & 5 \end{bmatrix}$ as in Example 55.

Example 118. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional). Note that we are lucky and already have an orthogonal basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is not invertible (because 0 is an eigenvalue) and therefore also cannot be orthogonal. Like any projection matrix, A is symmetric.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

(reflections) Suppose that M is the matrix for reflecting through the plane W in 3-space.

- The 1-eigenspace of M is W. (dimension 2)
- The -1-eigenspace of M is $W^\perp$. (dimension 1)

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get reflected to themselves. But those are precisely the vectors in the plane W (only vectors on the plane are unchanged by the reflection). On the other hand, the -1-eigenspace consists of those vectors $\boldsymbol{v}$ that get reflected to $-\boldsymbol{v}$ (the exact opposite direction). These are precisely the vectors orthogonal to the plane.

As in the case of projection matrices, because the eigenvalues are real and the eigenspaces are orthogonal, the reflection matrices are symmetric.

Comment. In this context, the line $W^\perp$ is often called the **normal line** of the plane W.

Example 119. Let A be the matrix for reflecting through the plane $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, -1$. The 1-eigenspace of A is W, and the -1-eigenspace is $W^{\perp}$.

In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

As in the previous example, $W^{\perp} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is invertible (because 0 is not an eigenvalue).

Like any reflection matrix, A is symmetric.

Finally, note that $A^2 = I$ (reflecting twice isn't doing anything), so that $A^{-1} = A$. It follows that A is orthogonal, because $A^{-1} = A = A^T$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. Similarly, a $n \times n$ matrix corresponds to a reflection (through a hyperplane) if and only if it has a $(n-1)$-dimensional 1-eigenspace and a 1-dimensional -1-eigenspace and these two spaces are orthogonal.

An alternative way of computing reflection matrices. Realize that, if $\boldsymbol{n}$ is the vector orthogonal to the plane (i.e. $\boldsymbol{n}$ is the normal vector of the plane), then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

We already observed that $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hence, the reflection of $\boldsymbol{v}$ is $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n}) = \boldsymbol{v} - 2\boldsymbol{n}\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{\boldsymbol{n}\cdot\boldsymbol{n}} = \boldsymbol{v} - 2\frac{\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{v}}{\boldsymbol{n}^T\boldsymbol{n}} = \left(I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}}\right)\boldsymbol{v}$.

Accordingly, the reflection matrix is $A = I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. In other words, we got A from subtracting 2 times the projection matrix onto $\boldsymbol{n}$ from I.

Notes for Lecture 23

Fri, 3/20/2026

Application: Linear differential equations

Example 120. (warmup) Solve the **differential equation** (DE) $y' = 2$.

Solution. From calculus, we know that the solutions are of the form $y(t) = 2t + C$.

Comment. To get a unique solution, we need to specify additional information, like an initial condition.

Example 121. (warmup) Solve the **initial value problem** (IVP) $y' = 2$, $y(0) = 1$.

Solution. This has the unique solution $y(t) = 2t + 1$.

Example 122. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Solution. $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

(exponential function) e^t is the unique solution to $y' = y$, $y(0) = 1$.

From here, it follows that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$.

The latter is the Taylor series for e^t at $t = 0$ that we have seen in Calculus II.

Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.

Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}t} \frac{t^3}{3!} = \frac{t^2}{2!}$.

Example 123. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Solution. Indeed, if $y(t) = Ce^{3t}$, then $y'(t) = 3Ce^{3t} = 3y(t)$.

Comment. It is important to realize that we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

Example 124. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Solution. As in the previous example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$.

Since $y(0) = Ce^0 = C = y_0$, we conclude that the unique solution to the IVP is $y(t) = e^{at} y_0$.

Comment. It looks silly to write $e^{at} y_0$ instead of $y_0 e^{at}$ here, but we will soon replace the number a with a matrix A, and in that case only $e^{At} y_0$ makes sense.

Example 125. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrr} y_1' &=& 2y_1 & & \\ y_2' &=& -y_1 & +3y_2 & +y_3 \\ y_3' &=& -y_1 & +y_2 & +3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form, this becomes

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The key idea will be to solve $\boldsymbol{y}' = A\boldsymbol{y}$ by introducing e^{At}.

Theorem 126. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

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Recall from Example 124 that the solution to $y' = ay$, $y(0) = y_0$ is $y(t) = e^{at} y_0$. Here, however, At is a matrix and so we need to make sense of the matrix exponential. Next time, we will define e^A by the familiar Taylor series for e^x.

Definition 127. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Why? As a consequence of this definition (which is the motivation for that definition in the first place),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e^{At} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[I + At + \frac{1}{2!}A^2 t^2 + \frac{1}{3!}A^3 t^3 + \cdots \right] \\ &= 0 + A + A^2 t + \frac{1}{2!}A^3 t^2 + \cdots = A e^{At}. \end{aligned}$$

Therefore, $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$ indeed solves the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

How to actually compute e^A? Well, this Taylor series involves the powers A^n of A. How would you compute, say, A^{100}? The answer is diagonalization!

Theorem 128. Suppose $A = PDP^{-1}$. Then, $e^A = Pe^D P^{-1}$.

Why? Recall that, if $A = PDP^{-1}$, then $A^n = PD^n P^{-1}$.

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots \right)P^{-1} = Pe^D P^{-1} \end{aligned}$$

Comment. By the same argument, if $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ for every "nice" function f. Here, "nice" means that f has a convergent Taylor series $f(x) = \sum_{n \geqslant 0} a_n x^n$.
More explicitly, if $A = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,P^{-1}$, then $f(A) = P\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))\,P^{-1}$.

Example 129. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

Example 130. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $At = \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix}$, $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Review.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

  Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

- If we have the diagonalization $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$ (and $e^{At} = Pe^{Dt}P^{-1}$).

- If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Comment. We only discuss **linear** differential equations (DEs). **Non-linear** DEs include $y' = y^2 + 1$ or the second-order equation $y'' = \sin(ty') + y$. The **order** of a DE indicates the highest occuring derivative.

We will see here how to solve those linear DEs which have constant coefficients (for instance, $y'' = \sin(t)y' + y$ is linear but the coefficients include $\sin(t)$ which is not constant). That is, the coefficients of y are constants, as opposed to functions (like $\sin(t)$) depending on t.

Example 131. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, we diagonalize:

  For $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$. (That's homework!)

- We can then compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned}
\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ 0 \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}
\end{aligned}$$

Comment. It is not necessary to compute $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$ (of course, you could do it, but that's more work).

Instead, recall that $A^{-1}\boldsymbol{b}$ is the unique solution to $A\boldsymbol{x} = \boldsymbol{b}$. Here, solving $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, we find $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Check. $\boldsymbol{y} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$ indeed solves the original problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} + 4e^{4t} \\ 4e^{4t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 + 1 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example 132. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} \boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$ has characteristic polynomial $-\lambda(1-\lambda) - 2 = (\lambda+1)(\lambda-2)$.

  Hence, the eigenvalues of A are $-1, 2$.

  The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} 2e^{-t} & -e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix}} \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2e^{-t} + e^{2t} \\ e^{-t} - e^{2t} \end{bmatrix} \end{aligned}$$

Check. Since it is simple to check, it would be almost criminal to not verify that $\boldsymbol{y}(0) = \begin{bmatrix} 2+1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Higher-order differential equations

Example 133. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Note. The "trick" of looking at the pair $\begin{bmatrix} y \\ y' \end{bmatrix}$ instead of a single function is what we used to translate the Fibonacci recurrence into a 2×2 system.

Alternatively. Instead of looking at the pair $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$ we could also consider the pair $\boldsymbol{y} = \begin{bmatrix} y' \\ y \end{bmatrix}$. In the latter case, the system becomes $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}$. Either choice is fine and it makes no difference for the computations.

Example 134. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 135. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Example 136. Suppose that $e^{Mt} = \frac{1}{10}\begin{bmatrix} e^t + 9e^{2t} & 3e^t - 3e^{2t} \\ 3e^t - 3e^{2t} & 9e^t + e^{2t} \end{bmatrix}$.

(a) Without doing any computations, determine M^n.

(b) What is M?

(c) Without doing any computations, determine the eigenvalues and eigenvectors of M.

Solution.

(a) Recall that $e^{Mt} = Pe^{Dt}P^{-1}$ while $M^n = PD^nP^{-1}$, provided that $M = PDP^{-1}$. The fact the formula for e^{Mt} features e^t and e^{2t}, means that the eigenvalues of M must be 1 and 2. Hence,

$$D = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix}, \quad D^n = \begin{bmatrix} 1 & \\ & 2^n \end{bmatrix}.$$

Therefore, we just need to replace e^t by $1^n = 1$ as well as e^{2t} by 2^n to get:

$$M^n = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2^n & 3 - 3 \cdot 2^n \\ 3 - 3 \cdot 2^n & 9 + 2^n \end{bmatrix}$$

(b) In particular, we see that the underlying matrix is $M = M^1 = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2 & 3 - 3 \cdot 2 \\ 3 - 3 \cdot 2 & 9 + 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$.

[Alternatively, we can find M by computing $\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} = Me^{Mt}$ and then setting $t = 0$.]

(c) The eigenvalues are 1 and 2.

Looking at the coefficients of e^t in the first column of e^{Mt}, we can see that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a 1-eigenvector.

[We can also look the second column of e^{Mt}, to obtain $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ which is a multiple and thus equivalent.]

Likewise, we find that $\begin{bmatrix} 9 \\ -3 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a 2-eigenvector.

The Jordan normal form

Note that we currently only know how to compute e^{At} when A is diagonalizable. Our next goal is to see how one can compute the matrix exponential for all matrices.

Example 137. Diagonalize, if possible, the matrix $A = \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}$.

Solution. The eigenvalues of A are $4, 4$.

However, the 4-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1-dimensional.

Hence, A is not diagonalizable.

Definition 138. A **λ-Jordan block** is a matrix of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$.

Note that if this matrix is $m \times m$, then its only eigenvalue is λ (repeated m times).

As in the previous example, the λ-eigenspace is 1-dimensional (which is as small as possible).

Theorem 139. (Jordan normal form) Every $n \times n$ matrix A can be written as $A = PJP^{-1}$, where J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}$$

with each J_i a Jordan block. J is called the **Jordan normal form** of A.

Up to the ordering of the Jordan blocks, the Jordan normal form of A is unique.

Comment. If A is diagonalizable, then J is just a usual diagonal matrix.

Review. Jordan normal form

Example 140. What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues $4, 4, 4$?

Solution. $\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$

The dimension of the 4-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you know from diagonalization).

Example 141.

(a) What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$?

(b) What are the possible Jordan normal forms of a 4×4 matrix with eigenvalues $3, 3, 3, 3$?

(c) What if the matrix is 5×5 and has eigenvalues $4, 4, 3, 3, 3$?

Solution.

(a) $\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 3 & 1 & \\ & 3 & \\ & & 3 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you known from diagonalization).

(b) Now, there are 5 possibilities:

$\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 4, 3, 2, 2, 1, respectively.

(c) $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$

Note that this is just all possible (namely, 3) Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$ combined with all possible (namely, 2) Jordan normal forms of a 2×2 matrix with eigenvalues $4, 4$. In total, that makes $3\cdot2=6$ possibilities.

Comment. Let $p(n)$ be the number of inequivalent Jordan normal forms of an $n\times n$ matrix with a single eigenvalue, n times repeated. We have seen that $p(2)=2$, $p(3)=3$, $p(4)=5$. Note that $p(n)$ is equal to the number of ways of writing n as an ordered sum of positive integers: for instance, $p(4)=5$ because $4=3+1=2+2=2+1+1=1+1+1+1$.

$p(n)$ is referred to as the **partition function** and, surprisingly, is a remarkably interesting mathematical object.

https://en.wikipedia.org/wiki/Partition_function_(number_theory)

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Example 142. (summary of small cases)

(a) There are 2 possible Jordan normal forms of a 2×2 matrix with eigenvalues λ,λ.

Namely. $\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$

(b) There are 3 possible Jordan normal forms of a 3×3 matrix with eigenvalues λ,λ,λ.

Namely. $\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$

(c) There are 5 possible Jordan normal forms of a 4×4 matrix with eigenvalues $\lambda,\lambda,\lambda,\lambda$.

Namely. $\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$

Example 143. What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues $3,3,7,7,7,7$?

Solution. There are $2\cdot 5 = 10$ possible Jordan normal forms for such a matrix:

$$\begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 &&& \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 &&& \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix}$$

Example 144. How many different Jordan normal forms are there in the following cases?

(a) A 8×8 matrix with eigenvalues $1,1,2,2,2,4,4,4$?

(b) A 11×11 matrix with eigenvalues $1,1,1,2,2,2,2,4,4,4,4$?

Solution.

(a) $2\cdot 3\cdot 3 = 18$ possible Jordan normal forms

(b) $3\cdot 5\cdot 5 = 75$ possible Jordan normal forms

The matrix exponential of matrices that are not diagonalizable

Review.

- Let A be $n\times n$. The matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Then, $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$.

Why? $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$

- If $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

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Example 145. The matrix exponential shares many other properties of the usual exponential:

- $e^A e^B = e^{A+B} = e^B e^A$ if $AB = BA$

  Why the condition $AB = BA$? By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$ In order to simplify that to

$$e^A e^B = \left(I + A + \frac{A^2}{2!} + \ldots\right)\left(I + B + \frac{B^2}{2!} + \ldots\right),$$

  we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- e^A is invertible and $(e^A)^{-1} = e^{-A}$

  Why? That actually follows from the previous property.

Example 146. Compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$.

Solution. Note that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{At} = I + At + \frac{t^2}{2!}A^2 + \ldots = I + At = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

Example 147. Compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$.

Solution. Note that $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$.

Hence, $e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots = I + At + \frac{1}{2}A^2 t^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ & 0 & t \\ & & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & t^2 \\ & 0 & 0 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix}$.

Example 148. Compute e^{At} for $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$.

Solution.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$. Note that $2I$ and N commute.

  Hence, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!}N^2 + \ldots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ & e^{2t} \end{bmatrix}$.

Advanced. Can you show that, similarly, $A^n = \begin{bmatrix} 2^n & n2^{n-1} \\ & 2^n \end{bmatrix}$?

Example 149. Solve the differential equation

$$\boldsymbol{y}' = \underbrace{\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}}_{A} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_0}$$

Solution. Repeating the work in the previous example, the solution to the differential equation is

$$\begin{aligned} \boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 \\ &= e^{2It+Nt}\boldsymbol{y}_0 \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= e^{2It}e^{Nt}\boldsymbol{y}_0 \qquad \text{(because } 2It \text{ and } Nt \text{ commute)} \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\left(I + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots\right)\boldsymbol{y}_0 \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}(I + Nt)\boldsymbol{y}_0 \qquad \text{(because } N^2 = \boldsymbol{0}) \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix}. \end{aligned}$$

Check. We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$.
Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

Comment. For applications, having solutions like $te^{\lambda t}$ or $t\cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.

Important comment. Note that we can immediately see from the solution that the original matrix A is not diagonalizable: there is a term te^{2t}, whereas in the diagonalizable case we would only see exponentials like e^{2t} by themselves.

In our upcoming discussion of complex numbers we will see that e^{2it} (here, $2i$ would be the eigenvalue) can be rewritten in terms of $\cos(2t)$ and $\sin(2t)$. Both of these are periodic and bounded, so that the same is true for every linear combination.

In that case, if the eigenvalue $2i$ was repeated in such a way that the matrix A is not diagonalizable, then we would get the functions $t\cos(2t)$ and $t\sin(2t)$ in our solutions. These, however, are not bounded! This phenomenon (getting solutions that are unbounded under the right/wrong circumstances) is called **resonance**.

https://en.wikipedia.org/wiki/Resonance

Understanding when resonance occurs is of crucial importance for practical applications.

Review. Jordan normal form

Rotation matrices

Example 150. Write down a 2×2 matrix Q for rotation by angle θ in the plane.

Comment. Why should we even be able to represent something like rotation by a matrix? Meaning that $Q\boldsymbol{x}$ should be the vector $\boldsymbol{x}$ rotated by θ. Recall from Linear Algebra I that every **linear map** can be represented by a matrix. Then think about why rotation is a linear map.

Solution. We can determine Q by figuring out $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the first column of Q) and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the second column of Q).

Since $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, we conclude that $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Comment. Note that we don't need previous knowledge of $\cos$ and $\sin$. We could have introduced these trig functions on the spot.

Comment. Note that it is geometrically obvious that Q is orthogonal. (Why?)

It is clear that $\left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|^2 = 1$. Noting that $\left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|^2 = \cos^2\theta + \sin^2\theta$, we have rediscovered Pythagoras.

Advanced comment. Actually, every orthogonal 2×2 matrix Q with $\det(Q) = 1$ is a rotation by some angle θ. Orthogonal matrices with $\det(Q) = -1$ are reflections.

Example 151. As in the previous example, let Q_θ be the 2×2 matrix for rotation by angle θ in the plane. What is $Q_\alpha Q_\beta$?

Solution. Note that $Q_\alpha Q_\beta \boldsymbol{x}$ first rotates $\boldsymbol{x}$ by angle β and then by angle α. For geometric reasons, it is obvious that this is the same as if we rotated $\boldsymbol{x}$ by $\alpha + \beta$. It follows that $Q_\alpha Q_\beta = Q_{\alpha+\beta}$.

Comment. This allows us to derive interesting trig identities:

$$\begin{aligned} Q_\alpha Q_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \ldots \\ \ldots & \ldots \end{bmatrix} \\ Q_{\alpha+\beta} &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \end{aligned}$$

It follows that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Comment. If we set $\beta = \alpha$, this simplifies to $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$, the double angle formula that you have probably used countless times in Calculus.

Comment. Similarly, we find an identity for $\sin(\alpha+\beta)$. Spell it out!

More on complex numbers

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

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Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at Example 89.

- The **absolute value** of the complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$.
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$.
  Note that $\|\boldsymbol{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\boldsymbol{z}}^T \boldsymbol{z}$.

Definition 152.

- For every matrix A, its **conjugate transpose** is $A^* = (\bar{A})^T$.
- The **dot product** (inner product) of complex vectors is $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^* \boldsymbol{w}$.
- A complex $n \times n$ matrix A is **unitary** if $A^* A = I$.

Comment. A^* is also written A^H (or $A^\dagger$ in quantum mechanics) and called the Hermitian conjugate.

Comment. For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

Comment. Unitary matrices are the complex version of orthogonal matrices. (A real matrix if unitary if and only if it is orthogonal.)

Example 153. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

Solution. $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

Example 154. What is $\frac{1}{2+3i}$?

Solution. $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$.

In general. $\frac{1}{z} = \frac{\bar{z}}{z\,\bar{z}} = \frac{\bar{z}}{|z|^2}$

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Review. Complex numbers

Example 155. (April Fools' Day!) Foul play with complex numbers:
$1=\sqrt{1}=\sqrt{(-1)(-1)}=\sqrt{-1}\sqrt{-1}=i i=-1.$

When using the principal square-root (which basically takes the positive root, that is, the one with positive real part), the rule $\sqrt{a b}=\sqrt{a}\,\sqrt{b}$ does not hold universally (so the trouble lies with the third equality). It does hold if, for instance, $a \geqslant 0$ or $b \geqslant 0$. Apparently, this trouble historically resulted in controversy around complex numbers, with some mathematicians rejecting them outright.

Example 156. (April Fools' Day!) What is the norm of the vector $\begin{bmatrix} 1 \\ i \end{bmatrix}$? Could it be zero?!

Solution. $\left\|\begin{bmatrix} 1 \\ i \end{bmatrix}\right\|=\sqrt{|1|^2+|i|^2}=\sqrt{1^2+1^2}=\sqrt{2}$

Comment. If we carelessly use $\left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|=\sqrt{a^2+b^2}$ (which works fine only when a and b are real numbers), then we would (incorrectly) get that the vector has norm 0.

Example 157. What is the norm of the vector $\begin{bmatrix} 4-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\|\begin{bmatrix} 4-i \\ 2+3i \end{bmatrix}\right\|=\sqrt{|4-i|^2+|2+3i|^2}=\sqrt{(4^2+(-1)^2)+(2^2+3^2)}=\sqrt{30}$

$\left\|\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}\right\|^2=\begin{bmatrix} 1+i & 2-3i \end{bmatrix}\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}=|1-i|^2+|2+3i|^2=2+13$. Hence, $\left\|\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}\right\|=\sqrt{15}$.

Example 158. True or false? Every $n\times n$ matrix A always has an eigenvector v.

Solution. True! Every $n\times n$ matrix always has exactly n eigenvalues (if we allow complex eigenvalues and count with repetition).

If λ is one of those eigenvalues, then the dimension of the λ-eigenspace is at least 1 (because $\det(A-\lambda I)=0$ so that $A\boldsymbol{v}=\lambda\boldsymbol{v}$ has nonzero solutions $\boldsymbol{v}$).

Example 159. (April Fools' Day!) Let A be an $n\times n$ matrix. By the previous example, we can find an eigenvector $\boldsymbol{v}$ and an eigenvalue λ such that $A\boldsymbol{v}=\lambda\boldsymbol{v}$.

Let us rewrite that as $A\boldsymbol{v}=\lambda I\boldsymbol{v}$ where I is the $n\times n$ identity matrix. Does it follow that $A=\lambda I$?

Solution. No, that is (of course) not a valid conclusion. One cannot cancel a vector $\boldsymbol{v}$ in this way.

Recall, more generally for matrices, that in order to conclude from $AB=CB$ that $A=C$ we need extra information on B such as B being invertible.

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