Midterm #2

MATH 316 — Linear Algebra II
Monday, April 13, 2026

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 35 points in total. You need to show work to receive full credit.

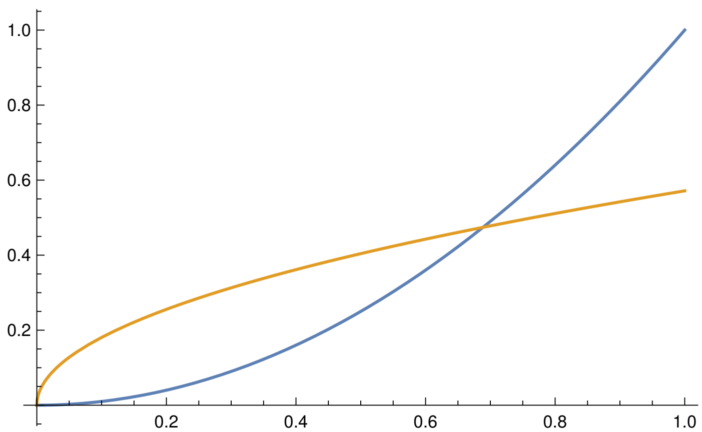
Good luck!

Problem 1. (5 points) Find the best approximation (in the L^2 sense) of $f(x) = x^2$ on the interval $[0,1]$ using a function of the form $y(x) = a\sqrt{x}$.

Solution. The orthogonal projection of $f\colon [0,1] \to \mathbb{R}$ onto $\operatorname{span}\{\sqrt{x}\}$ is

$$\frac{\langle f, \sqrt{x}\rangle}{\langle \sqrt{x}, \sqrt{x}\rangle}\sqrt{x} = \frac{\int_0^1 f(t)\sqrt{t}\,\mathrm{d}t}{\int_0^1 t\,\mathrm{d}t}\sqrt{x} = 2\sqrt{x}\int_0^1 t^{5/2}\mathrm{d}t = 2\sqrt{x}\left[\frac{1}{7/2}t^{7/2}\right]_0^1 = \frac{4}{7}\sqrt{x}.$$

Not an impressive approximation, but the best possible given the constraints:



Problem 2. (4 points) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + 4y_1 \\ y_2'' &= y_1' - y_2' - 3y_2 \end{aligned} \qquad y_1(0) = 1,\ y_1'(0) = 4,\ y_2(0) = 0,\ y_2'(0) = 2$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 5 & 2 \\ 0 & -3 & 1 & -1 \end{bmatrix}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}.$$

Armin Straub
straub@southalabama.edu

Problem 3. (8 points) Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

- If $A = \begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$, then $\det(A - \lambda I) = (-1-\lambda)(3-\lambda) - 5 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$. The eigenvalues are $4, -2$.

  The 4-eigenspace $\operatorname{null}\left(\begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  The -2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & \\ & -2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & \\ & e^{-2t} \end{bmatrix}}_{\begin{bmatrix} e^{4t} & -5e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix}} \underbrace{\frac{1}{6} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}} = \frac{1}{6} \begin{bmatrix} 5e^{4t} - 5e^{-2t} \\ 5e^{4t} + e^{-2t} \end{bmatrix} \end{aligned}$$

Problem 4. (4 points) Fill in the blanks.

(a) Let A be a 4×4 matrix for orthogonally projecting onto a 3-dimensional subspace.

Then $\det(A) =$ ______, and the eigenvalues (indicate if repeated) of A are ______.

(b) If A is the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then $A = PDP^{-1}$ with $D =$ ______ and $P =$ ______. Moreover, $\det(A) =$ ______.

Solution.

(a) $\det(A) = 0$ and the eigenvalues of A are $1, 1, 1, 0$.

(b) Note that $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal and that the normal direction is spanned by $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$.

Hence, we can choose $P = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

We have $\det(A) = \det(D) = -1$.

Comment. If we normalize the columns of P so that P is orthogonal (which means $P^{-1} = P^T$) then we would get a diagonalization of the type $A = PDP^T$.

Armin Straub
straub@southalabama.edu

Problem 5. (6 points) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) Find an explicit (Binet-like) formula for a_n.

(b) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ are $2, -1$.

Hence, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 1$, $a_1 = 2\alpha_1 - \alpha_2 = 8$.

Solving, we find $\alpha_1 = 3$ and $\alpha_2 = -2$ so that, in conclusion, $a_n = 3 \cdot 2^n - 2(-1)^n$.

(b) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$.

Problem 6. (8 points) Fill in the blanks.

(a) The norm of the vector $\boldsymbol{v} = \begin{bmatrix} 2-3i \\ 1 \end{bmatrix}$ is $\|\boldsymbol{v}\| =$ ______.

(b) If A is a projection matrix, then $A^{2026} =$ ______. If B is a reflection matrix, then $B^{2026} =$ ______.

(c) If A has eigenvalue 4, then $3A$ has eigenvalue ______, A^2 eigenvalue ______, and A^T eigenvalue ______.

(d) If $A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$, then $A^n =$ ______ and $e^{At} =$ ______.

(e) An example of a 2×2 matrix with eigenvalue $\lambda = 4$ that is not diagonalizable is ______.

(f) If $N^3 = \mathbf{0}$, then $e^{Nt} =$ ______.

(g) How many different Jordan normal forms are there in the following cases?

- A 5×5 matrix with eigenvalues $2, 2, 2, 4, 4$? ______
- A 7×7 matrix with eigenvalues $4, 4, 4, 4, 5, 5, 6$? ______

Solution.

(a) $\left\| \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} \right\| = \sqrt{2^2 + 3^2 + 1} = \sqrt{14}$

Armin Straub
straub@southalabama.edu

(b) If A is a projection matrix, then $A^{2026} = A$. (Because $A^2 = A$.)

If B is a reflection matrix, then $B^{2026} = I$. (Because $B^2 = I$.)

(c) If A has eigenvalue 4, then $3A$ has eigenvalue $3 \cdot 4 = 12$, A^2 eigenvalue $4^2 = 16$, and A^T eigenvalue 4.

(d) If $A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$, then $A^n = \begin{bmatrix} (-2)^n & \\ & 4^n \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{-2t} & \\ & e^{4t} \end{bmatrix}$.

(e) An example of a 2×2 matrix with eigenvalue $\lambda = 4$ that is not diagonalizable is $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$. (This is a Jordan block!)

(f) If $N^3 = \mathbf{0}$, then $e^{Nt} = I + Nt + \frac{1}{2}N^2t^2$.

(g) $3 \cdot 2 = 6$ and $5 \cdot 2 \cdot 1 = 10$ different Jordan normal forms.

Armin Straub
straub@southalabama.edu

(extra scratch paper)