Example 179. (extra) Can we generalize the previous example by replacing 2 with x?
That is, we are now interested in the sums $s(n) = 1 + x + x^2 + \ldots + x^n$.

Mimic previous direct approach. $x s(n) = x(1 + x + x^2 + \ldots + x^n) = x + x^2 + \ldots + x^{n+1} = s(n) - 1 + x^{n+1}$.
Hence, $(x-1)s(n) = x^{n+1} - 1$, and we have found:

$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}$	**(geometric sum)**

Sigma notation. Instead of $1 + x + x^2 + \ldots + x^n$ we will begin to write $\sum_{k=0}^{n} x^k$.

Geometric series. We can let $n \to \infty$ to get $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, provided that $|x| < 1$.

Example 180. (Homework) Prove the formula for geometric sums using induction.

Example 181. (sum of squares) For all integers $n \geqslant 1$, $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Write $t(n) = 1^2 + 2^2 + \ldots + n^2$.
We use induction on the claim $t(n) = \frac{n(n+1)(2n+1)}{6}$.

- The base case $(n = 1)$ is that $t(1) = 1$. That's true.
- For the inductive step, assume the formula holds for some value of n.
  We need to show the formula also holds for $n + 1$.

$$\begin{aligned} t(n+1) &= t(n) + (n+1)^2 \\ \text{(using the induction hypothesis)} \quad &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)}{6}[2n^2 + n + 6n + 6] \\ &= \frac{(n+1)}{6}(n+2)(2n+3) \end{aligned}$$

  This shows that the formula also holds for $n + 1$.

By induction, the formula is true for all integers $n \geqslant 1$. □

Example 182. Observe the following connection with our sums and integrals from calculus:

- $\int_0^n x \mathrm{d}x = \frac{n^2}{2}$ versus $\sum_{x=0}^{n} x = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \text{lower order terms}$
- $\int_0^n x^2 \mathrm{d}x = \frac{n^3}{3}$ versus $\sum_{x=0}^{n} x^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \text{lower order terms}$
- $\int_0^n x^3 \mathrm{d}x = \frac{n^4}{4}$ versus $\sum_{x=0}^{n} x^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4}{4} + \text{lower order terms}$

The connection makes sense: the integrals give areas below curves, and the sums are approximations to these areas (rectangles of width 1).

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Example 183. (Riemann hypothesis) The **Riemann zeta function** $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ converges (for real s) if and only if $s > 1$.

The divergent series $\zeta(1)$ is the harmonic series, and $\zeta(p)$ is often called a p-series in Calculus II.

Comment. Euler achieved worldwide fame by discovering and proving that $\zeta(2) = \frac{\pi^2}{6}$ (and similar formulas for $\zeta(4), \zeta(6), \ldots$).

For complex values of $s \neq 1$, there is a unique way to "analytically continue" this function. It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, …. The **Riemann hypothesis** claims that all other zeroes of $\zeta(s)$ lie on the line $s = \frac{1}{2} + a\sqrt{-1}$ ($a \in \mathbb{R}$). A proof of this conjecture (checked for the first 10,000,000,000,000 zeroes) is worth $1,000,000.

http://www.claymath.org/millennium-problems/riemann-hypothesis

The connection to primes. Here's a vague indication that $\zeta(s)$ is intimately connected to prime numbers:

$$\begin{aligned}
\zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \ldots\right)\cdots \\
&= \frac{1}{1 - 2^{-s}} \frac{1}{1 - 3^{-s}} \frac{1}{1 - 5^{-s}} \cdots \\
&= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}
\end{aligned}$$

This infinite product is called the Euler product for the zeta function. If the Riemann hypothesis was true, then we would be better able to estimate the number $\pi(x)$ of primes $p \leqslant x$.

More generally, certain statements about the zeta function can be translated to statements about primes. For instance, the (non-obvious!) fact that $\zeta(s)$ has no zeros for $\operatorname{Re} s = 1$ implies the prime number theorem that we discussed earlier.

http://www-users.math.umn.edu/~garrett/m/v/pnt.pdf