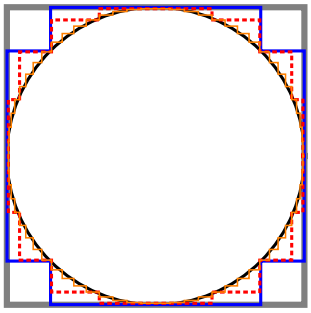
(Halloween scare!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi/4$, the area of the circle.)

See below for a "solution" to the Halloween scare...

Example 132. Python Let us redo Example 131 by implementing the trapezoidal rule.

```
>>> def trapezoidal_rule(f, a, b, n):
        h = (b - a) / n
        integral = f(a) + f(b)
        for i in range(1,n):
            integral += 2*f(a + i*h)
        return h/2*integral

>>> def f(x):
        return 1/x
```

Comment. Writing `x += y` is a useful and common short alternative to `x = x + y`.

Choosing `n` to be 2 and 4 is equivalent to $h = \frac{3-1}{2} = 1$ and $h = \frac{3-1}{4} = \frac{1}{2}$ and so we get the same values as in Example 131:

```
>>> trapezoidal_rule(f, 1, 3, 2)

   1.1666666666666665

>>> trapezoidal_rule(f, 1, 3, 4)

   1.1166666666666667
```

As expected, further increasing n produces better approximations:

```
>>> [trapezoidal_rule(f, 1, 3, 10**n) for n in range(1,4)]

   [1.1015623265623264, 1.0986419169811203, 1.0986125849642736]
```

Indeed, the following convincingly illustrates that the error in the trapezoidal rule is $O(h^2)$.

```
>>> from math import log

>>> [trapezoidal_rule(f, 1, 3, 10**n) - log(3) for n in range(1,6)]

   [0.0029500378942166616, 2.9628313010565677e-05, 2.962961638264261e-07,
   2.9629636522088276e-09, 2.962430301067798e-11]
```

However, note that our computer had to work pretty hard to get the final approximation, because that entailed computing about 10^5 values. We clearly should use a higher order method in order to compute to higher accuracy. One option is to do what we did in the last part of Example 131.

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Simpson's rule

Let us spell out what happens in general when we proceed as in the last part of Example 131. We start with $T(h)$, the trapezoidal rule applied with step size h, which is given by

$$T(h) = \frac{h}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

Then, since $T(h)$ is an approximation of order $N = 2$, the Richardson extrapolation

$$S(h) := \frac{2^N T(h) - T(2h)}{2^N - 1} = \frac{4}{3}T(h) - \frac{1}{3}T(2h)$$

is an approximation of higher order, known as **Simpson's rule**. It takes the following form:

(Simpson's rule) Suppose n is even. The following is an approximation of order 4:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Proof. To see this, note that the trapezoidal approximations $T(h)$ and $T(2h)$ are given by

$$\begin{aligned} T(h) &= \frac{h}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)], \\ T(2h) &= h[f(x_0) + 2f(x_2) + \cdots + 2f(x_{n-2}) + f(x_n)]. \end{aligned}$$

Here, we need n to be even so that $T(2h)$ uses the points x_0, x_2, ..., x_{n-2}, x_n. Therefore, the Richardson extrapolation is

$$\begin{aligned} S(h) = \frac{4}{3}T(h) - \frac{1}{3}T(2h) &= \frac{h}{3}[2f(x_0) + 4f(x_1) + \cdots + 4f(x_{n-1}) + 2f(x_n)] \\ &\quad -\frac{h}{3}[f(x_0) + 2f(x_2) + \cdots + 2f(x_{n-2}) + f(x_n)] \\ &= \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)], \end{aligned}$$

where the right-hand side is Simpson's rule. That this is an approximation of order 4 follows because the error of $T(h)$ is an even function in h, so that the order of $S(h)$ increases to 4 (rather than the otherwise expected 3). □

Alternative approach. We can also proceed like for the trapezoidal rule, except that we use quadratic instead of linear interpolations on each segment. More precisely, starting with equally spaced nodes $x_0, x_1, ..., x_n$ with n even, we can interpolate $f(x)$ on each segment $[x_{i-1}, x_{i+1}]$, with i odd, by a quadratic function $p(x)$. The integral on that segment works out to be

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\mathrm{d}x \approx \int_{x_{i-1}}^{x_{i+1}} p(x)\mathrm{d}x \overset{\text{work}}{=} \frac{h}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})],$$

where we wrote $h = (b-a)/n$ for the distance between nodes. Adding together the integrals over all segments, each node x_i, with i odd, will show up once with the above factor of $4h/3$ while x_i, with i even, (except x_0 and x_n) will show up twice with a factor of $h/3$. We thus again get Simpson's integration rule.

Comment. The above rule is often called **Simpson's 1/3 rule**. There is also **Simpson's 3/8 rule** which is derived similarly but is based on a cubic (instead of a quadratic) interpolation; it thus requires an additional node (the resulting error term is of the same order but about half).

Similar to Theorem 130, one can show that the error in Simpson's rule is as follows:

Theorem 133. (Simpson's rule with error term) If f is C^4 smooth, then

$$\int_a^b f(x)\mathrm{d}x = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(x_n)] - \frac{(b-a)}{180}f^{(4)}(\xi)h^4$$

for some $\xi \in [a,b]$.

Comment. In the alternative approach above, we constructed Simpson's rule so that quadratic polynomials would be integrated without error; the error term tells us that, in fact, cubic polynomials are also integrated without error. In other words, the error term is a pleasant surprise!

($\pi = 4$, "solution")

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\mathrm{perimeter}(c_n) \to \mathrm{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as we learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a,b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \int_a^b \sqrt{1+f_n'(x)^2}\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2\mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.