Euler's method applied to e^x

Example 141. Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with 4 steps. In particular, what is the approximation for $y(1)$?

Comment. Of course, the real solution is $y(x) = e^x$. In particular, $y(1) = e \approx 2.71828$.

Solution. The step size is $h = \frac{1-0}{4} = \frac{1}{4}$. We apply Euler's method with $f(x, y) = y$:

$$\begin{aligned}
x_0 &= 0 & y_0 &= 1 \\
x_1 &= \frac{1}{4} & y_1 &= y_0 + f(x_0, y_0)h = 1 + \frac{1}{4} = \frac{5}{4} = 1.25 \\
x_2 &= \frac{1}{2} & y_2 &= y_1 + f(x_1, y_1)h = \frac{5}{4} + \frac{5}{4} \cdot \frac{1}{4} = \frac{5^2}{4^2} = 1.5625 \\
x_3 &= \frac{3}{4} & y_3 &= y_2 + f(x_2, y_2)h = \frac{5^2}{4^2} + \frac{5^2}{4^2} \cdot \frac{1}{4} = \frac{5^3}{4^3} \approx 1.9531 \\
x_4 &= 1 & y_4 &= y_3 + f(x_3, y_3)h = \frac{5^3}{4^3} + \frac{5^3}{4^3} \cdot \frac{1}{4} = \frac{5^4}{4^4} \approx 2.4414
\end{aligned}$$

In particular, the approximation for $y(1)$ is $y_4 \approx 2.4414$.

Comment. Can you see that, if instead we start with $h = \frac{1}{n}$, then we similarly get $y_i = \frac{(n+1)^i}{n^i}$ for $i = 0, 1, \ldots, n$? In particular, $y(1) \approx y_n = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$. Do you recall how to derive this final limit?

Example 142. (cont'd) Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with n steps for several values of n. In each case, what is the approximation for $y(1)$?

Solution. Since the real solution is $y(x) = e^x$ so that, in particular, the exact solution is $y(1) = e \approx 2.71828$. We proceed as we did in Example 141 in the case $n = 4$ and apply Euler's method with $f(x, y) = y$:

$$\begin{aligned}
x_{n+1} &= x_n + h \\
y_{n+1} &= y_n + h \underbrace{f(x_n, y_n)}_{=y_n} = (1+h)y_n
\end{aligned}$$

We observe that it follows from $y_{n+1} = (1+h)y_n$ that $y_n = (1+h)^n y_0$. Since $y_0 = 1$ and $h = \frac{1-0}{n} = \frac{1}{n}$, we conclude that

$$x_n = 1, \quad y_n = \left(1 + \frac{1}{n}\right)^n.$$

[For instance, for $n = 4$, we get $x_4 = 1$, $y_4 = \left(\frac{5}{4}\right)^4 \approx 2.4414$ as in Example 141.]

In particular, our approximation for $y(1)$ is $\left(1 + \frac{1}{n}\right)^n$.

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Here are a few values spelled out:

$$\begin{aligned}
n=1: &\quad \left(1+\frac{1}{n}\right)^n = 2 \\
n=4: &\quad \left(1+\frac{1}{n}\right)^n = 2.4414... \\
n=12: &\quad \left(1+\frac{1}{n}\right)^n = 2.6130... \\
n=100: &\quad \left(1+\frac{1}{n}\right)^n = 2.7048... \\
n=365: &\quad \left(1+\frac{1}{n}\right)^n = 2.7145... \\
n=1000: &\quad \left(1+\frac{1}{n}\right)^n = 2.7169... \\
n\to\infty: &\quad \left(1+\frac{1}{n}\right)^n \to e = 2.71828...
\end{aligned}$$

We can see that Euler's method converges to the correct value as $n \to \infty$. On the other hand, we can see that it doesn't converge impressively fast. That is why, for serious applications, one usually doesn't use Euler's method directly but rather higher-order methods derived from the same principles (such as Runge–Kutta methods).

Interpretation. Note that we can interpret the above values in terms of compound interest. We start with initial capital of $y(0)=1$ and we are interested in the capital $y(1)$ after 1 year if we receive interest at an annual rate of 100%:

- If we receive a single interest payment at the end of the year, then $y(1)=2$ (case $n=1$ above).
- If we receive quarterly interest payments of $\frac{100\%}{4}=25\%$ each, then $y(1)=(1.25)^4=2.441...$ (case $n=4$).
- If we receive monthly interest payments of $\frac{100\%}{12}=\frac{1}{12}$ each, then $y(1)=2.6130...$ (case $n=12$).
- If we receive daily interest payments of $\frac{100\%}{365}=\frac{1}{365}$ each, then $y(1)=2.7145...$ (case $n=365$).

It is natural to wonder what happens if interest payments are made more and more frequently. Well, we already know the answer! If interest is compounded continuously, then we have e in our bank account after one year.